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The relationship of IA-central subgroup with other subgroups

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Abstract

Let G be a group. Ghumde and Ghate [7] in 2015 introduced the IA-central subgroup S(G) and they proved that if G is a group with G/S(G) finite, then so is IA(G) and G'. The IA-central subgroup is located between the absolute center and the center of the group. In this paper, we study the conditions in which S(G) is equal to each of these two subgroups. We also state the conditions for the equality of S(G) with G and when S(G) is non-trivial. At the end of this paper, we provide a converse to Ghumde and Ghate [7] theorem.

Keywords: IA-group, commutator subgroup, IA-central subgroup, semi-complete group, absolute center subgroup

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1 Introduction

The center of a group and their subgroups have interesting properties, Hence the subgroups have been the idea of many researchers articles. Let G be a group and p be a prime. Let us denote by Z(G), G', exp(G), $\pi(G)$, d(G), r(G), T(G), Hom(G, H), Aut(G) and Inn(G), respectively the centre, the commutator subgroup, the exponent, the set of primes dividing the order of G, the rank, the torsion-free rank, the torsion subgroup of G, the group of homomorphisms of G into an abelian group H, the full automorphism group and the inner automorphisms. Let $G^{p^n} = \langle g^{p^n} \mid g \in G\rangle$. Also,

$$L(G) = \left\{g \in G \;\middle|\; g^{-1}\alpha(g) = 1,\ \forall\ \alpha \in Aut(G)\right\},$$
$$IA(G) = \left\{\alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) \in G',\ \forall\ g \in G\right\},$$
$$IA_Z(G) = \left\{\alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) \in G',\ \alpha(z) = z,\ \forall\ g \in G,\ \forall\ z \in Z(G)\right\},$$

are the absolute center subgroup, the group of all automorphisms of G which induce identity map on G/G' and the group of those IA-automorphisms which fix the centre elementwise, respectively.
A metabelian group is a group whose commutator subgroup is abelian. A metacyclic group is a group G having a cyclic normal subgroup N such that the quotient G/N is also cyclic. A homocyclic group is a direct product of one or more pairwise isomorphic cyclic groups.
Let G be a finite group and N be non-trivial proper normal subgroup of G. The pair (G; N) is called a Camina pair if $xN \subseteq x^G$ for all $x^G \in G \setminus N$ where x^G denotes the conjugacy class of x in G. A group G

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is called a Camina group if (G, G') is a Camina pair. So if G is a Camina group then $G' = [x, G]$ for all $x \in G \setminus G'$.

On the lines of results of Schur and Hegarty, Ghumde and Ghate [7] in 2015 introduced the S(G) subgroup as follows:

$$S(G) = \{g \in G \mid g^{-1}\alpha(g) = [g, \alpha] = 1,\ \alpha \in IA(G)\}.$$

They showed that for a finite p-group G, S(G) is non-trivial and $L(G) \trianglelefteq S(G) \trianglelefteq Z(G)$. Also, if G/S(G) is finite, then so is G' and IA(G).

In 2020, Azhdari [4] named the subgroup S(G), $L_{G'}(G)$ and achieved the following results:

1) If G be a nilpotent group of class 2, then

   a) $G' \leqslant S(G)$ and thus G/S(G) is abelian.

   b) $exp\Big(T\big(G/S(G)\big)\Big) \Bigm| exp\big(T(G')\big)$.

   c) If G is a torsion-free group, then so are G/Z(G) and G/S(G).

   d) If G is a torsion group, then so is G/S(G).

2) Let G be a finite non-abelian p-group of class 2 and $exp(G') = p^n$, then

   a) $S(G) = G'G^{p^n}$.

   b) IA(G)=Inn(G) if and only if G' is cyclic and $Z(G) = S(G) = G'G^{p^n}$.

3) Let G be a finitely generated nilpotent group of class 2 with finite cyclic commutator, $G' = \langle b \rangle$. Let $G/Z(G) = \langle \overline{x_1} \rangle \times \cdots \times \langle \overline{x_d} \rangle$ where $\langle \overline{x} \rangle$ denotes xZ(G). Assign a skew-symmetric matrix $A_G = (a_{ij})$ to G where a_{ij} is defined by the equation $[x_i, x_j] = b^{ij}$ for all $1 \leq i, j \leq d$. Then IA(G)=Inn(G) if and only if one of the following conditions holds:

   a) G' is finite and Z(G)=S(G).

   b) G' is infinite, $r\big(Z(G)\big) = 1$ and $det(A_G) = 1$.

2 Main results

In this section, we study the conditions in which S(G) is equal to L(G) and Z(G). Also, We investigate conditions where S(G)=G and S(G) is non-trivial. The last theorem states the converse of Ghumde and Ghate [7] theorem and prove it.

The following proposition clearly states the conditions of equality of S(G) and L(G).

Proposition 2.1. *For a group G, S(G)=L(G) if*

1) G be a complete group, i.e. $G = G'$.

2) $[G : G'] = 2$, because then IA(G)=Aut(G) [7].

In the following, we will study the conditions that S(G)=Z(G). According to the definition of S(G) and Z(G), this relationship will be established if IA(G)=Inn(G). Therefore, we consider groups that have this condition. In other words, G is semi-complete.

Proposition 2.2. *For a group G, S(G)=Z(G) if*

1) G be the wreath product of a infinite cyclic group by another [6].

2) G be a free metabelian group of rank 2 [12].

3) *G be a free product of two non-trivial abelian groups [1].*

4) *G' be cyclic and $IA(G) = IA_Z(G)$ [2].*

5) *G be a finite p-group of class 2 where $Hom(G/G', G') \cong G/Z(G)$ [2].*

6) *G be a finite p-group of class 2, G' is cyclic and $Hom\big(G/Z(G), G'\big) \cong Hom(G/G', G')$ [2].*

7) *G be a metabelian 2-generated finite p-group where $|G| = |G'|^2|Z(G)|$ [2].*

8) *G be a finite 2-generated p-group of class at most 4 where $[G : Z(G)] = |G'|^2$ [2].*

9) *G be a finite 2-generated p-group(or nilpotent group) of class 2 [2]([14]).*

10) *G be a non-abelian group of order p^4, $G/Z(G) \cong C_p \times C_p$ and $G/G' \cong C_p \times C_{p^2}$ [2].*

11) *G be a non-abelian group of order p^5 and one of the following conditions holds[2]:*

    a) *$Z(G) \cong C_p$ and $G/G' \cong C_p \times C_{p^2}$.*
    b) *$G/Z(G) \cong C_p \times C_p$ and $G/G' \cong C_{p^2} \times C_{p^2}$.*
    c) *$G/Z(G) \cong C_p \times C_p$ and $G/G' \cong C_{p^3} \times C_p$.*

12) *G be a extra-special group [8].*

13) *G be a p-group with a cyclic maximal subgroup and $|G| = 2^3$ or*
$$G \cong M(p^n) = \langle x, y \mid x^{p^{n-1}} = y^p = 1,\ y^{-1}xy = x^{1+p^{n-2}} \rangle \text{ [8].}$$

14) *G be a finite p-group of class 2, G' is cyclic and $Z(G) = G'G^{p^n}$ where $|G'| = p^n$ [14].*

15) *G be a finite p-group such that G' is cyclic, $\big(G, Z(G)\big)$ is a Camina pair and G is isomorphic to a central product $A * X_{p^3}^{*n}$ for some $n \geq 0$ and an odd prime p where*

    a) *A is a 2-generator subgroup which is either a metacyclic group or $A = \langle a\rangle\langle b\rangle\langle c\rangle$, $[a, c] = [b, c] = 1$, $[a, b] = cb^{p^k}$ where $k \geq 1$,*
    b) *X_{p^3} is non-abelian finite p-group of order p^3 and exponent p and*
    c) *X^{*n} is the iterated central product defined by $X^{*n} = X * X^{*(n-1)}$ with $X^{*1} = X$ [14].*

16) *G be an abelian group [3].*

17) *G ba a finite p-group of class 2 and G' is cyclic and G is Camina. [3].*

18) *G be a finitely generated nilpotent group of class 2 and G' is cyclic. if G' is infinite then $det(A_G) = 1$ Which was introduced in (3) of Azhdari theorems at the end of Section 1 [3].*

19) *G be a finitely generated nilpotent group of class 2 with infinite cyclic commutator, $r\big(Z(G)\big) = 1$ and $det(A_G) = 1$ Which was introduced in (3) of Azhdari theorems at the end of Section 1 [3].*

20) *G be a finite non-abelian p-group of class 2, $exp(G') = p^n$, G' is cyclic and $Z(G) = S(G) = G'G^{p^n}$ [4].*

21) *G be a finitely generated nilpotent group of class 2 with cyclic commutator and one of the following conditions holds[4]:*

    a) *G' is finite and Z(G)=S(G).*
    b) *G' is infinite, $r\big(Z(G)\big) = 1$ and $det(A_G) = 1$ Which was introduced in (3) of Azhdari theorems at the end of Section 1.*

22) *G be a finite p-group of class 2, $G/Z(G) \cong \prod_{i=1}^{m} C_{p^{a_i}}$ and $G/G' \cong \prod_{j=1}^{n} C_{p^{b_j}}$ where $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ are positive integers. G' is cyclic, $d\big(G/Z(G)\big) = d(G/G')$ and either $G/Z(G)$ is homocyclic or $a_i = a_1$ for $1 \leq i \leq t$ and $b_i = a_i$ for $t \leq i \leq n$ where t is the smallest positive integer such that $b_t \geq a_1$ [13].*

23) *G be a finitely generated nilpotent group of class 2, $G/Z(G) \cong A \times \mathbb{Z}^a$, $G/G' \cong B \times \mathbb{Z}^b$, $G' \cong C \times \mathbb{Z}^c$, where $A \cong \prod_{i=1}^{d} \prod_{j=1}^{m_i} C_{p_i^{\alpha_{ij}}}$, $\pi(A) = \{p_1, p_2, \ldots, p_d\}$, $\alpha_{11} \geq \alpha_{12} \geq \cdots \geq \alpha_{1m_1}$, $\alpha_{21} \geq \alpha_{22} \geq \cdots \geq \alpha_{2m_2}$, $\ldots$, $\alpha_{d1} \geq \alpha_{d2} \geq \cdots \geq \alpha_{dm_d}$, $B \cong \prod_{i=1}^{e} \prod_{j=1}^{n_i} C_{p_i^{b_{ij}}}$, $\pi(B) = \{p_1, p_2, \ldots, p_e\}$, $\beta_{11} \geq \beta_{12} \geq \cdots \geq \beta_{1n_1}$, $\beta_{21} \geq \beta_{22} \geq \cdots \geq \beta_{2n_2}$, $\ldots$, $\beta_{e1} \geq \beta_{e2} \geq \cdots \geq \beta_{en_e}$, A, B, C are respective torsion parts and a, b, c are respective torsion-free ranks of G/Z(G), G/G' and G'. Also one of the following conditions holds[13]:*

- i) *G is torsion-free, G' is cyclic and $r\big(G/Z(G)\big) = r(G/G')$.*
- ii) *G is not torsion-free and one of the following four conditions holds:*
  - a) *G is finite, $m_i = n_i$ for $1 \leq i \leq d$, G' is cyclic and for each $1 \leq i \leq d$ either $\big(G/Z(G)\big)_{p_i}$ is homocyclic or $\alpha_{i1} = \alpha_{it_i}$ for $i1 \leq it_i \leq i(r_i - 1)$ and $\beta_{it_i} = \alpha_{it_i}$ for $ir_i \leq it_i \leq in_i$ where ir_i is the smallest positive integer between $i1$ and in_i such that $\beta_{ir_i} < \alpha_{i1}$.*
  - b) *$G' \cong \mathbb{Z}$ and $r\big(G/Z(G)\big) = r(G/G')$.*
  - c) *G' is torsion, G/G' is torsion-free and $A \cong C^b$.*
  - d) *G/Z(G) and $G' \cong C \times \mathbb{Z}$ are mixed groups, G/G' is torsion-free, $A \cong C^b$ and $r\big(G/Z(G)\big) = r(G/G')$.*

24) *G be a finite p-group such that $Z(G) \subseteq G'$, $IA\big(G/Z(G)\big) = Inn\big(G/Z(G)\big)$ and $C_{Aut(G)}\big(Inn(G)\big) = Z\big(Inn(G)\big)$ [10].*

There may be other conditions for equalization S(G)=Z(G). The following examples shows that every IA-automorphism is not necessarily inner automorphism and the equality S(G)=Z(G) does not always hold.

Example 2.3. 1) Bachmuth [5] has shown that the IA-group of a free metabelian group of rank two is equal to its inner automorphism group. He has also shown that this is not the case when the rank is larger than two.

2) Taheri et. al. [15] considered the group $G = \langle a, b, x \mid [a, x] = [b, x] = 1,\ [a, b] = x^k,\ k \neq 1\rangle$ and they showed that IA_Z-automorphism α defined by $\alpha(a) = ax^k$, $\alpha(b) = bx^k$, $\alpha(x) = x$ is a non-inner automorphism, but $G' = \langle x^k\rangle$, $Z(G) = S(G) = \langle x\rangle$.

3) Ghumde and Ghate [8] considered the group $G = H \times K$ where groups H and K given by $H = \langle x, y \mid x^4 = y^2 = 1,\ yxy^{-1} = x^{-1}\rangle$, $K = \langle z, w \mid z^4 = w^2 = 1,\ wzw^{-1} = z^{-1}\rangle$. The map $f :\ H \times K \longrightarrow H \times K$ which is defined as follows $x \longrightarrow xz^2$, $y \longrightarrow yz^2$, $z \longrightarrow zx^2$, $w \longrightarrow wx^2$ is an IA_Z-automorphism which is not an inner automorphism. Here $G' = \langle x^2, z^2\rangle$, $Z(G) = \langle x, z\rangle$ and $S(G) = e$.

Theorem 2.4. *Let G be an abelian group, then S(G)=G.*

Proof. Because G is abelian, then $G' = e$, $IA(G) = \langle 1\rangle$. hence S(G)=G. □

Now, we study the conditions in which S(G) is non-trivial. In the above theorem, we saw that for abelian groups S(G)=G, therefore, in the following, we consider non-abelian groups.

Theorem 2.5. *Let G be a group and $H \leqslant G$, then $H \leqslant S(G)$ if*

1) *$Aut(G) = C_{Aut(G)}(H)$. The converse of this part is also true, i.e.*
$$H \leqslant S(G) \iff Aut(G) = C_{Aut(G)}(H).$$

2) *G be a finite group and H be a characteristic subgroup of prime order p such that p be the smallest prime divisor of $|Aut(G)|$.*

3) Aut(G) be a perfect group and H be a cyclic characteristic subgroup of G.

Proof. Given that $L(G) \leqslant S(G)$, the proof easily follow from [11] lemma 2.4(iv), corollary 3.5 and 3.7 respectively. □

Theorem 2.6. *Let G be a group, Aut(G) be a finite p-group and H be a finite characteristic subgroup of G such that $p||H|$, then $H \cap S(G) \neq \langle 1 \rangle$.*

Proof. Because H is a characteristic subgroup of G, then this equivalence relation yields a partition of H and each cell in the partition arising from an equivalence relation is equivalence class. According to lemma 2.5 [11], there is $1 \neq h_0 \in H$ element such that the equivalence class is of order 1. So we have $\alpha(h_0) = h_0$, for every $\alpha \in Aut(G)$. Thus $1 \neq h_0 \in S(G) \cap H$ and this completes the proof. □

Corollary 2.7. *If G be a finite group such that Aut(G) is a p-group, then $S(G) \neq \langle 1 \rangle$.*

Theorem 2.8 (MacHale[9])**.** *Let G be a finite group such that Aut(G) is nilpotent. If G is not cyclic of odd order, then G contains a non-trivial element which is left fixed by every automorphism of G.*

Corollary 2.9. *Let G be a finite group such that Aut(G) is nilpotent, then $S(G) \neq \langle 1 \rangle$.*

Theorem 2.10 (The converse of Ghumde and Ghate [7] theorem)**.** *Let G be an arbitrary group. If G' and IA(G) are both finite, then so is G/S(G).*

Proof. We define $C_G(\alpha) = \{g \in G \mid [g, \alpha] = 1\}$, for every $\alpha \in IA(G)$. Then $[G : C_G(\alpha)]$ is finite, since G' is finite. But $S(G) = \bigcap_{\alpha \in IA(G)} C_G(\alpha)$ and since this is a finite intersection, it follows that G/S(G) is finite. □

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