# An upper bound for the group of IA-automorphisms

Sara Barin[1]
Department of Mathematics, University of Birjand, Iran

Mohammad mehdi Nasrabadi
Department of Mathematics, University of Birjand, Iran

**Abstract**

Following Bachmuth [1], we call an automorphism on group G an IA-automorphism if and only if it preserves all cosets of $G'$. Ghumde and Ghate [4] have been obtained the lower and upper bounds on the number of IA-automorphisms of a fnitely generated group. Also, Ghumde and Ghate [3] in 2015 introduced the IA-central subgroup S(G) and Ivar(G) group. In this paper, we derive an upper bound for IA(G) in terms of G/S(G) order.

**Keywords:** IA-group, commutator subgroup, IA-central subgroup, Ivar(G), upper bound
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## 1 Introduction

Let G be a group. Let us denote by Z(G), $G'$, Hom(G, H), Aut(G) and Inn(G), respectively the centre, the commutator subgroup, the group of homomorphisms of G into an abelian group H, the full automorphism group and the inner automorphisms.

On the lines of results of Schur [6] and Hegarty [5], Ghumde and Ghate [3] in 2015 introduced the S(G) subgroup as follows:

$$S(G) = \left\{ g \in G \mid g^{-1}\alpha(g) = [g, \alpha] = 1,\ \alpha \in IA(G) \right\}.$$

They showed that for a finite p-group G, S(G) is non-trivial. Afterward, they introduced Ivar(G) subgroup as follows:

$$Ivar(G) = \left\{ \alpha \in IA(G) \mid g^{-1}\alpha(g) \in S(G),\ \forall g \in G \right\}.$$

The purpose of this article is to derive upper bounds for $|IA(G)|$ in terms of $|G/S(G)|$.

## 2 Main results

In this section, we state our main theorem and prove it.

**Theorem 2.1** (Main theorem). *Let G be an arbitrary group and $|G/S(G)| = n$, then*

$$|IA(G)| \leq n! \times n^{(n-1)^2}.$$

[1]speaker

Before proving the theorem, we need the following lemma and theorem.

**Lemma 2.2** (Lemma 1.2[5]). *Let $G$ be any group and $N \subseteq Z(G)$ a subgroup of $G$ such that $[G:N]$ is finite. Let*

$$G = Ng_1 \cup Ng_2 \cup \cdots \cup Ng_k$$

*be a complete set of cosets of $N$ in $G$. Consider a fixed set of coset representatives $\{g_1 = 1, g_2, \ldots, g_k\}$. For each ordered pair $(i, j) \in \{1, \ldots, k\} \times \{1, \ldots, k\}$. Define the element $n_{ij}$ of $N$ as follows:*

$$n_{ij} = g_i g_j g_l^{-1},$$

*where $Ng_ig_j = Ng_l$. Let $\varphi$ be an automorphism of $N$ leaving every $n_{ij}$ fixed, then $\varphi$ can be extended to an automorphism of $G$.*

**Theorem 2.3** (Kulikov theorem[2]). *Let $G$ be an infinite abelian group which is not torsion-free, then $G$ has a direct summand of the form $C_{p^k}$, $1 \leq k \leq \infty$ for some prime $p$.*

**Proof of main theorem.** IA(G) act on G/S(G) as following

$$\begin{aligned} f: \frac{G}{S(G)} \times IA(G) &\longrightarrow \frac{G}{S(G)} \\ (gS(G), \alpha) &\longmapsto g^{\alpha}S(G) = \alpha(g)S(G), \end{aligned}$$

whose kernel is as follows

$$\begin{aligned} N &= \left\{\alpha \in IA(G) \mid f\big(gS(G), \alpha\big) = gS(G),\ \forall\, g \in G\right\}, \\ &= \left\{\alpha \in IA(G) \mid g^{\alpha}S(G) = gS(G),\ \forall\, g \in G\right\}, \\ &= \left\{\alpha \in IA(G) \mid g^{-1}g^{\alpha} \in S(G),\ \forall\, g \in G\right\}, \\ &= Ivar(G). \end{aligned}$$

From the proof of theorem 3.10 [3], IA(G)/Ivar(G) is isomorphic to a subgroup of permutation group $S_n$, so we have

$$|IA(G)| \leq n!|Ivar(G)|. \tag{1}$$

Again, from the proof of theorem 3.10 [3], there exists a monomorphism between elements $\alpha$ of Ivar(G) and elements $\alpha^*$ of Hom(G/S(G),S(G)) as follows:

$$\begin{aligned} \sigma: Ivar(G) &\longrightarrow Hom\Big(\frac{G}{S(G)}, S(G)\Big) \\ \alpha &\hookrightarrow \alpha^* \end{aligned}$$

where

$$\begin{aligned} \alpha^*: \frac{G}{S(G)} &\longrightarrow S(G) \\ gS(G) &\longmapsto g^{-1}\alpha(g), \quad \forall\, g \in G. \end{aligned}$$

We define

$$S_0 = \Big\langle \alpha^*\Big(\frac{G}{S(G)}\Big) \Bigm| \alpha \in Ivar(G)\Big\rangle.$$

Because G/S(G) is finite, so is Ivar(G) by the proof of theorem 3.10 [3], therefore $S_0 \leqslant S(G)$ is finite too. By Kulikov's theorem, $S_0$ can be embedded in a torsion direct summand of S(G). Let S be any such direct summand. By Lemma 2.2, S is finite and

$$d(S) \leq n^2 - 2n + 1 = (n-1)^2$$

(using the notation of Lemma 2.2, $n_{i1} = n_{1i} = 1$, $1 \leq i \leq n$, since $g_1 = 1$). Then every $\alpha^*$ is in Hom(G/S(G),S), for every $\alpha \in Ivar(G)$. Now, suppose that

$$\Big|\frac{G}{S(G)}\Big| = n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}. \qquad (2)$$

Hence using the fundamental theorem of finitely generated abelian groups, we have

$$\frac{G/S(G)}{\big(G/S(G)\big)'} \cong C_{p_1^{b_1}} \times C_{p_2^{b_2}} \cdots C_{p_k^{b_k}}, \qquad \forall\, 1 \leq i \leq k: \quad 0 \leq b_i \leq a_i. \qquad (3)$$

On the other hands, without loss of generality, $|S|$ is divisible only by primes among the set $\{p_1, p_2, \ldots, p_k\}$. Since S is finite and abelian, by the fundamental theorem of finitely generated abelian groups, we have

$$S \cong \prod_{i=1}^{k} \prod_{j=1}^{(n-1)^2} C_{p_i^{c_j^i}}, \qquad (4)$$

such that

$$\forall\, 1 \leq i \leq k,\ 1 \leq j \leq (n-1)^2: \quad c_j^i \in \mathbb{N} \cup \{0\}.$$

Since for any group G and an abelian group A, $Hom(G, A) \cong Hom(G/G', A)$, thus by (2), (3) and (4), we have

$$\begin{aligned} |Ivar(G)| &= \Big|Hom\Big(\frac{G}{S(G)}, S\Big)\Big| \\ &= \Big|Hom\Big(\frac{G/S(G)}{\big(G/S(G)\big)'}, S\Big)\Big| \\ &= \Big|Hom\Big(\prod_{i=1}^{k} C_{p_i^{b_i}}, \prod_{i=1}^{k}\prod_{j=1}^{(n-1)^2} C_{p_i^{c_j^i}}\Big)\Big| \\ &= \prod_{i=1}^{k}\prod_{j=1}^{(n-1)^2} p_i^{min(b_i, c_j^i)} \\ &\leq \prod_{i=1}^{k}\prod_{j=1}^{(n-1)^2} p_i^{a_i} \\ &= n^{(n-1)^2}. \end{aligned} \qquad (5)$$

Now, by (1) and (5) we have

$$|IA(G)| \leq n! \times n^{(n-1)^2}.$$

## References


[1] S. Bachmuth, *Automorphisms of free metabelian groups*, Trans. Amer. Math. Soc. 118, (1965), 93-104.

[2] L. Fuchs, *Abelian Groups*, Pergammon Press, Elmsford, NY, (1960).

[3] R. G. Ghumde, and S. H. Ghate, *IA-automorphisms of p-groups, finite polycyclic groups and other results*, Matematicki Vesnik, 67 (2015), 194-200.

[4] R. G. Ghumde, and S. H. Ghate, *Bounds of the group of IA-automorphisms*, Eurasian Mathematical Journal, 7(4), (2016), 30-45.

[5] P. V. Hegarty, *The absolute centre of a group*, J. Algebra, 169 (1994), 929-935.

[6] I. Schur, *Uber die Darstellung der Endlichen Gruppen durch Gebrochene Lineare Substitutionen*, J. Reine Angew. Math. 127, (1904), 20-50.

Email: s.barin@birjand.ac.ir
Email: mnasrabadi@birjand.ac.ir