Some basic results of Ivar(G) automorphisms group

Sara Barin[1]
Department of Mathematics, University of Birjand, Iran

Mohammad mehdi Nasrabadi
Department of Mathematics, University of Birjand, Iran

Abstract

Bachmuth [2] in 1965 defined an IA-automorphism of a group G as an automorphism of G which induces the identity automorphism on G/G'. Hegarty [3] in 1994 introduced the absolute center L(G) and absolute centeral automorphisms $Aut_l(G)$. On the similar lines, Ghumde and Ghate [4] in 2015 introduced the IA-central subgroup S(G) and Ivar(G) group. In this paper, we prove some properties of Ivar(G) and give the groups that are isomorphic with it.

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1 Introduction

Let G be a group and p be a prime. Let us denote by Z(G), G', Hom(G, H), Aut(G) and Inn(G), respectively the centre, the commutator subgroup, the group of homomorphisms of G into an abelian group H, the full automorphism group and the inner automorphisms. Also,

$$\begin{aligned} L(G) &= \left\{ g \in G \;\middle|\; g^{-1}\alpha(g) = 1,\ \forall\, \alpha \in Aut(G) \right\}, \\ IA(G) &= \left\{ \alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) \in G',\ \forall\, g \in G \right\}, \\ Aut_c(G) &= \left\{ \alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) \in Z(G),\ \forall\, g \in G \right\}, \\ Aut_l(G) &= \left\{ \alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) \in L(G),\ \forall\, g \in G \right\}, \end{aligned}$$

On the lines of results of Schur and Hegarty, Ghumde and Ghate [4] in 2015 introduced the S(G) subgroup as follows:

$$S(G) = \left\{ g \in G \mid g^{-1}\alpha(g) = [g, \alpha] = 1,\ \alpha \in IA(G) \right\}.$$

They showed that for a finite p-group G, S(G) is non-trivial and

$$L(G) \trianglelefteq S(G) \trianglelefteq Z(G). \tag{1}$$

Afterward, they introduced Ivar(G) subgroup as follows:

$$Ivar(G) = \left\{ \alpha \in IA(G) \mid g^{-1}\alpha(g) \in S(G),\ \forall g \in G \right\}.$$

At the end of the article, they also proved that if G is a group with G/S(G) finite, then so is IA(G) and G'. Also, G' finite implies that Iver(G) is finite.

[1]speaker

2 Main results

In this section, we first study some basic properties of S(G) and Ivar(G), afterwards by using this properties we give several result about the relationship of Ivar(G) with other automorphisms and various groups. In second subsection, we terminate this article with some theorems of groups that are isomorphisms with Ivar(G).

2.1 Preliminary properties

First we prove $\emptyset \neq Ivar(G) \trianglelefteq Aut(G)$, but before proving it, we need the following proposition.

Proposition 2.1. *Let G be a group, then S(G) is a characteristic subgroup of G.*

Proof. we know that $S(G) \leqslant G$ and $S(G) \leqslant Z(G) \overset{ch}{\leqslant} G$. We prove that $S(G) \overset{ch}{\leqslant} Z(G)$, because then $S(G) \overset{ch}{\leqslant} G$ by 2.11.12 [5]. Let $\beta \in Aut\big(Z(G)\big)$ and $s \in S(G)$, we show that $\beta(s) \in S(G)$.

By IA(G) definition, $[\beta(s), \alpha] = \big(\beta(s)\big)^{-1}\alpha\big(\beta(s)\big) \in G'$ for every $\alpha \in IA(G)$. As $\beta(s) \in Z(G)$, so $S(G) \leqslant Z(G) \leqslant G'$. For abelian group $Z(G)$, $Aut\big(Z(G)\big) = Aut_c\big(Z(G)\big)$, therefore $\beta \in Aut\big(Z(G)\big) = Aut_c\big(Z(G)\big)$. Since $\beta(s) \in Z(G) \leqslant G'$ and the central automorphisms fix G' pointwise, so $\beta(s) = s \in S(G)$. ☐

Theorem 2.2. *Let G be a group, then Ivar(G) is a non-trivial and normal subgroup of Aut(G).*

Proof. For every arbitrary group G, the identity automorphism is an element of Ivar(G), therefore $Ivar(G) \neq \emptyset$. According to the previous proposition, it is clear that Iver(G) is a subgroup of Aut(G), so we only prove that the Iver(G) is normal in Aut(G).

Let $\beta \in Aut(G)$ and $\alpha \in Ivar(G)$ be arbitrary. we show that $\beta^{-1}\alpha\beta \in Ivar(G)$. For every $g \in G$, we have $\beta^{-1}(g)\alpha\big(\beta(g)\big) \in S(G)$, thus there exists $s_0 \in S(G)$ such that $\beta^{-1}(g)\alpha\big(\beta(g)\big) = s_0$. Now,

$$g^{-1}\big(\beta^{-1}\alpha\beta(g)\big) = g^{-1}\beta^{-1}\big(\alpha\beta(g)\big) = g^{-1}\beta^{-1}\big(\beta(g)\beta^{-1}(g)\alpha\beta(g)\big) = g^{-1}\beta^{-1}\big(\beta(g)s_0\big) = g^{-1}g\beta^{-1}(s_0)$$
$$= \beta^{-1}(s_0) \in \beta^{-1}\big(S(G)\big).$$

Because $S(G) \overset{ch}{\leqslant} G$, then $\beta^{-1}\big(S(G)\big) = S(G)$. thus $g^{-1}\big(\beta^{-1}\alpha\beta(g)\big) \in S(G)$ and the result follows. ☐

Remark 2.3. According to the relation (1) and $Aut_l(G)$, $Aut_c(G)$ and Ivar(G) definitions we have:

$$Aut_l(G) \leqslant Ivar(G) \leqslant Aut_c(G). \tag{2}$$

Lemma 2.4. *Let G be an abelian group, then $Ivar(G) = \langle 1 \rangle$.*

Proof. According to Ivar(G) and IA(G) definitions, if G be an abelian group, then $IA(G) = \langle 1 \rangle$, so $Ivar(G) = \langle 1 \rangle$. ☐

Proposition 2.5. *Let G be a group.*

1) *If $Inn(G) \leqslant Ivar(G)$, then Inn(G) is abelian.*

2) *Every automorphism in Ivar(G) preserves the commutator subgroup G' elementwise.*

Proof. Obviously, this results are follow from remark 2.3 and properties of central automorphisms. ☐

Theorem 2.6. *Let G be a finite p-group, then Ivar(G) is also a p-group.*

Proof. By theorem 2.6 [4], if G is a finite p-group, then IA(G) is also a p-group. Since $Ivar(G) \leqslant IA(G)$, the result follows. ☐

Theorem 2.7. *Let G be a group.*

1) If Ivar(G)=Inn(G), then G is a nilpotent group of class at most 2.

2) If Ivar(G)=Aut(G), then G is a nilpotent group of class at most 2.

Proof. 1) Since Ivar(G)=Inn(G), then $x^{-1}\theta_g(x) \in S(G)$ for every $g \in G$, $\theta_g \in Inn(G)$. Thus $x^{-1}g^{-1}xg \in S(G)$, hence $G' \leqslant S(G)$.

2) $Aut(G) = Ivar(G) \leqslant Aut_c(G)$, therefore $Aut_c(G) = Aut(G)$. Now the result follows from properties of central automorphisms. □

2.2 The groups that are isomorphic with Ivar(G)

Adney and Yen [1] proved that if G is a purely non-abelian finite group, then there exists a bijection between $Aut_c(G)$ and $Hom\big(G, Z(G)\big)$. Following their result, we have the following theorem. The proof is done in a similar way to the Adney and Yen theorem, so we refrain from mentioning it.

Theorem 2.8. *Let G be a purely non-abelian finite group, then there exists a bijection between $Ivar(G)$ and $Hom\big(G, S(G) \cap G'\big)$.*

Since for any group G and an abelian group A

$$Hom(G, A) \cong Hom(\frac{G}{G'}, A), \tag{3}$$

the above theorem will yield the following result.

Theorem 2.9. *Let G be a purely non-abelian finite group, then there exists a bijection between $Ivar(G)$ and $Hom\big(G/G', S(G) \cap G'\big)$.*

Corollary 2.10. *Let G be a purely non-abelian finite group, then*

$$|Ivar(G)| = |Hom\big(G, S(G) \cap G'\big)| = |Hom\big(\frac{G}{G'}, S(G) \cap G'\big)|.$$

Proposition 2.11. *Let G be a finite group such that $S(G) \leqslant G'$, then*

$$Ivar(G) \cong Hom\big(G, S(G)\big) \cong Hom\big(\frac{G}{G'}, S(G)\big).$$

Proof. Using (3), we need to show that $Ivar(G) \cong Hom\big(G/G', S(G)\big)$. We consider map $\psi: \ Ivar(G) \longrightarrow Hom\big(G/G', S(G)\big)$ defined by $\psi(\alpha) = f_\alpha$ where $f_\alpha(gG') = g^{-1}\alpha(g)$, for all $g \in G$. We prove that ψ is an isomorphism. Since $x^{-1}f_\alpha(x) \in S(G)$ for each $x \in G$ and Ivar(G) fixes G' pointwise, ψ is well defined and $f_\alpha \in Hom\big(G/G', S(G)\big)$. Since $S(G) \leqslant G'$ and Ivar(G) fixes G' pointwise, it follows that ψ is a homomorphism. It is clear that ψ is injective. To see that ψ is surjective, let $\beta \in Hom\big(G/G', S(G)\big)$. Define $\alpha: \ G \longrightarrow G$ by $\alpha(g) = g\beta(gG')$ for all $g \in G$. Now it is clear that α is a monomorphism. Because G is finite and α is one to one, α is surjective, hence $\alpha \in Ivar(G)$. From the definition ψ, it is obvious that $f_\alpha = \beta$ and this complete the proof. □

Proposition 2.12. *Let G be a group, Then $Ivar(G) \cong Hom\big(G/S(G), S(G) \cap G'\big)$. In particular, Ivar(G) is an abelian group.*

Proof. Consider the map $\alpha^*: \ G/S(G) \longrightarrow S(G) \cap G'$ defined by $\alpha^*\big(gS(G)\big) = g^{-1}\alpha(g)$, for all $g \in G$ and each $\alpha \in Ivar(G)$. Since every automorphism in Ivar(G) acts trivially on S(G), α^* is a well-defined homomorphism of G/S(G) into $S(G) \cap G'$. Now it is easy to check $\psi: \ Ivar(G) \longrightarrow Hom\big(G/S(G), S(G) \cap G'\big)$ defined by $\psi(\alpha) = \alpha^*$, for any $\alpha \in Ivar(G)$ is an isomorphism.

For second part, we know that $S(G) \cap G' \leqslant S(G)$ is an abelian group, so $\alpha\beta\big(gS(G)\big) = \beta\alpha\big(gS(G)\big)$ for each $\alpha, \beta \in Hom\big(G/S(G), S(G) \cap G'\big)$ and $g \in G$. Thus, $Hom\big(G/S(G), S(G) \cap G'\big)$ is an abelian group. Now, the result follows by first part. □

Corollary 2.13. *Let G be a finite group.*

1) If $\big(|G/S(G)|, |S(G) \cap G'|\big) = 1$, then $Ivar(G) = \langle 1 \rangle$.

2) If $G/S(G)$ is an abelian group and $Ivar(G) = \langle 1 \rangle$, then $\big(|G/S(G)|, |S(G) \cap G'|\big) = 1$.

Proof. 1) We assume by way of contradiction that $Ivar(G) \neq \langle 1 \rangle$, i.e. there exist a non-trivial homomorphism $\alpha : \ G/S(G) \longrightarrow \big(S(G) \cap G'\big)$. because $Ivar(G) \neq \langle 1 \rangle$, so $Ker(\alpha) \lhd G/S(G)$. By first isomorphism theorem $\dfrac{G/S(G)}{Ker(\alpha)} \cong Im(\alpha)$ and by Lagrange's theorem $|Im(\alpha)| \mid |S(G) \cap G'|$. Therefore, we have $\big(|G/S(G)|, |Im(\alpha)|\big) = 1$, since otherwise if $\big(|\dfrac{G}{S(G)}|, |Im(\alpha)|\big) = n$, then

$$n \mid |G/S(G)| \qquad and \qquad n \mid |Im(\alpha)|,$$

thus

$$n \mid |G/S(G)| \qquad and \qquad n \mid |S(G) \cap G'|,$$

contrary to the assumption. Also,

$$\begin{aligned} & \Big|\frac{G/S(G)}{Ker(\alpha)}\Big| = |Im(\alpha)| \\ \Longrightarrow \quad & \Big|\frac{G}{S(G)}\Big| = |Ker(\alpha)||Im(\alpha)| \\ \Longrightarrow \quad & \Big(\Big|\frac{G}{S(G)}\Big|, |Im(\alpha)|\Big) = |Im(\alpha)| \neq 1. \end{aligned}$$

This contradiction completes the proof.
2) Proposition 2.12 implies that $Hom\big(G/S(G), S(G) \cap G'\big) = \langle 1 \rangle$. Using the fundamental theorem of finitely generated abelian groups and the property that $Hom(C_m, C_n) \cong C_{(m,n)}$, we obtain $\big(|G/S(G)|, |S(G) \cap G'|\big) = 1$. □

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Email: s.barin@birjand.ac.ir
Email: mnasrabadi@birjand.ac.ir