Results on some energies of the Zero-divisor graph of the commutative ring $\mathbb{Z}_n$

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Abstract

Let $Z(R)$ be the set of zero-divisors of a commutative ring R with non-zero identity and $Z^*(R) = Z(R) \setminus \{0\}$. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple graph whose vertex set in $Z^*(R)$ and two vertices u and v are adjacent if and only if $uv = vu = 0$.
In this paper, we investigate some energies of graphs $\Gamma(R)$ and line graph $\Gamma(R)$ for the commutative ring $\mathbb{Z}_n$.

Keywords: Commutative ring, Zero-divisor graph, line graph, minimum edge dominating energy, Laplacian energy

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1 Introduction

Throughout this paper, all graphs are finite, undirected and simple. Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$. The set $N_G(u) = \{v \in V | uv \in E\}$ is called the neighborhood of vertex $u \in V$ in graph G. The number of edges incident to vertex u in G is denoted $deg_G(u) = d(u)$. The isolated vertex and pendant vertex are the vertices with degrees zero and 1 in graph G, respectively.
The adjacency matrix of G, $A(G) = (a_{ij})$ is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E$ and $a_{ij} = 0$ otherwise. The eigenvalues of graph G are the eigenvalues of its adjacency matrix $A(G)$ [11]. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the graph G. In [8], the energy of a graph G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. The edge energy of a graph G, denoted by $EE(G)$, is defined as the sum of the absolute values of eigenvalues of $A(L_G)$ [4]. The line graph L_G of G is the graph that each vertex of it represents an edge of G and two vertices of L_G are adjacent if and only if their corresponding edges are incident in G [11].
Let $D(G)$ be the diagonal matrix of order n whose (i, i)-entry is the degree of the vertex v_i of the graph G. Then the matrices $L(G) = D(G) - A(G)$ and $L^+(G) = D(G) + A(G)$ are the Laplacian matrix and the signless Laplacian matrix, respectively, of the graph G. If $\mu_1, \mu_2, \dots, \mu_n$ and $\mu_1^+, \mu_2^+, \dots, \mu_n^+$ are, respectively, the eigenvalues of the matrices $L(G)$ and $L^+(G)$, then the Laplacian energy of G is defined as [9]

$$LE = LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|,$$

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and the signless Laplacian energy is defined as follows [10]

$$LE^+ = LE^+(G) = \sum_{i=1}^{n} |\mu_i^+ - \frac{2m}{n}|.$$

More about Laplacian and signless Laplacian energies and energy of a line graph can be seen in [9, 10, 5, 6, 18].

A subset D of V is the dominating set of graph G if every vertex of $V \setminus D$ is adjacent to some vertices in D. Any dominating set with minimum cardinality is called a minimum dominating set [12]. In [20], the minimum dominating energy of graph G, denoted by $E_D(G)$, is introduced as the sum of the absolute values of eigenvalues of the minimum dominating matrix, $A_D(G)$ whose is a matrix obtained from the adjacency matrix and the dominating set D as following

$$A_D(G) := (a_{ij}) = \begin{cases} 1 & \text{if } v_iv_j \in E \\ 1 & \text{if } i = j \textit{ and } v_i \in D \\ 0 & \textit{otherwise} \end{cases}$$

Let G be a simple graph with edge set $\{e_1, e_2, \ldots, e_m\}$. A set F of edges in G is the edge dominating set if every edge in $E \setminus F$ is adjacent to at least one edge in F. The edge domination number, denoted by γ', is the minimum the cardinalities of the edge dominating sets of G [7]. Note that F is the minimum edge dominating set of G or the minimum dominating set of L_G. The minimum edge dominating matrix of G is the $m \times m$ matrix defined by $A_F(G) := (a_{ij})$ in which

$$A_F(G) := (a_{ij}) = \begin{cases} 1 & \text{if } e_i \textit{ and } e_j \textit{ are adjacent}, \\ 1 & \text{if } i = j \textit{ and } e_i \in F, \\ 0 & \textit{otherwise}. \end{cases}$$

The minimum edge dominating energy of G is introduced and studied in [1] as following

$$EE_F(G) = \sum_{i=1}^{m} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of $A_F(G)$. Note the minimum edge dominating energy of graph G is a minimum dominating energy for its line graph L_G. For more results about the minimum dominating energy of a graph and its line graph, see [20, 1, 15, 16, 17, 13].

Let R be a ring and $Z(R)$ denotes the set of all zero-divisors of R. The zero-divisor graph of R is a simple graph $\Gamma(R)$ with vertex set $Z(R) \setminus \{0\}$ such that distinct vertices x and y are adjacent if and only if $xy = 0$ [2]. In this paper, we consider the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ of the commutative ring $\mathbb{Z}_n$ of residue classes modulo a positive integer. We are interested in investigating some energies of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ and the line graph of $\Gamma(\mathbb{Z}_n)$.

Throughout this paper, K_n and S_n denote a complete graph and a star graph of order n, respectively. We shall use $K_{p,q}$ to denote the complete bipartite graph.

2 Preliminaries

In this section, we state some previous results that will be used in the next section.

Lemma 2.1. [15] *Let G be a graph of the order n with m edges. If F is the minimum edge dominating set of graph G with cardinality k, then*

$$EE_F(G) \leq 4m - 2n + k.$$

Lemma 2.2. [15] *Let G be a connected graph with n vertices and $m(\geq n)$ edges. Then*

$$EE_F(G) \geq 4(m - n + s) + 2p,$$

where p and s are the number of pendant and isolated vertices in G.

Lemma 2.3. [15] *Let G be a graph of order n with $m \geq 1$ edges and the independence number α. Then*

$$EE_F(G) \geq 2(E(G) - 2n + 2\alpha),$$

where $E(G)$ is the graph energy of G.

Lemma 2.4. [20] *The minimum dominating energy of a complete graph K_n of order $n \geq 2$ is*

$$E_D(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}.$$

Lemma 2.5. [1] *For $n \geq 3$, the minimum edge dominating energy of a star graph S_n is*

$$EF_F(S_n) = (n-3) + \sqrt{n^2 - 4n + 8}.$$

Lemma 2.6. [14] *Let G be an r-regular graph. Then, r is an eigenvalue of G. In addition, if λ is an eigenvalue of G then, $r - \lambda$ is the eigenvalue of the Laplacian graph $L(G)$.*

Lemma 2.7. [15] *Let G be a simple graph and L_G the line graph of G. If F is the minimum edge dominating set with $|F| = k$, then*

$$EE_F(G) \leq EE(G) + k.$$

Lemma 2.8. [23] *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph where $n = p^2$ and $p \geq 5$ any prime number. Then, the edge energy of $\Gamma(\mathbb{Z}_n)$ is equal to $2p^2 - 10p + 8$.*

Lemma 2.9. [23] *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph in which $n = pq$ for some distinct primes $p > 2$ and $q > 2$. Then, $EE(\Gamma(\mathbb{Z}_{pq})) = 4pq - 8(p+q) + 16$.*

Lemma 2.10. [3] *For any connected graph G of even order n, $\gamma' = \frac{n}{2}$ if and only if G is isomorphic to K_n or $K_{\frac{n}{2},\frac{n}{2}}$.*

3 Main Results

In this section, we study energies of graph $\Gamma(\mathbb{Z}_n)$ such as the edge energy, the minimum dominating energy, the minimum edge dominating energy, the Laplacian energy and the signless Laplacian energy of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$. If $n = p$ where p is a prime, then graph $\Gamma(\mathbb{Z}_n)$ has no edges. Because $\mathbb{Z}_n$ is a field and has no non-zero zero-divisors. Assume that $n = p^2$ where p is a prime. So, $Z^*(\mathbb{Z}_{p^2}) = \{p, 2p, \ldots, (p-1)p\}$ and for any $x, y \in Z^*(\mathbb{Z}_{p^2})$, $xy = 0$. Clearly, the zero-divisor graph $\Gamma(\mathbb{Z}_{p^2})$ is a complete graph of the order $p-1$ and size $\dfrac{(p-1)(p-2)}{2}$.

Theorem 3.1. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph with $n = p^2$ for a prime p. Then,*

$$LE(\Gamma(\mathbb{Z}_{p^2})) = LE^+(\Gamma(\mathbb{Z}_{p^2})) = 2(p-2).$$

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_{p^2})$. Since $\Gamma(\mathbb{Z}_{p^2})$ is a complete graph of the order $p-1$, thus its eigenvalues are $p-2$ with multiplicity 1 and -1 with multiplicity $p-2$. Using Lemma 2.6, the spectrum of Laplacian matrix and signless Laplacian matrix of $\Gamma(\mathbb{Z}_{p^2})$ are $spec\big(L(G)\big) = \big\{(0)^{[1]}, (p-1)^{[p-2]}\big\}$ and $spec\big(L^+(G)\big) = \big\{2(p-2)^{[1]}, (p-3)^{[p-2]}\big\}$, respectively. By putting $n = p-1$ and $m = \frac{(p-1)(p-2)}{2}$ in the definition Laplacian and signless Laplacian energies, we get

$$\begin{aligned} LE(G) &= \sum_{i=1}^{p-1} \left|\mu_i - \frac{2m}{n}\right| \\ &= \big|0 - (p-2)\big| + (p-2)\big|(p-1) - (p-2)\big| \\ &= 2(p-2). \end{aligned}$$

and

$$\begin{aligned} LE^+(G) &= \sum_{i=1}^{p-1} \left|\mu_i^+ - \frac{2m}{n}\right| \\ &= \big|2(p-2) - (p-2)\big| + (p-2)\big|(p-3) - (p-2)\big| \\ &= 2(p-2). \end{aligned}$$

□

Theorem 3.2. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph with $n = p^2$ where p is a prime. If D and F are the minimum dominating and the minmum edge dominating sets of graph $\Gamma(\mathbb{Z}_n)$, respectively, such that $|F| = k$, then*

i) for $p \geq 2$, $E_D\big(\Gamma(\mathbb{Z}_n)\big) = (p-3) + \sqrt{p^2 - 4p + 8}$.

ii) for $p \geq 5$, $2(p-1)(p-4) \leq EE_F\big(\Gamma(\mathbb{Z}_n)\big) \leq \frac{(p-1)(4p-15)}{2}$.

Proof. We suppose that G is the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ of the commutative ring $\mathbb{Z}_n$ for $n = p^2$ where p is a prime.

i) Since G is a complete graph of the order $p-1$, using Lemma 2.4 we have

$$E_D(K_{p-1}) = ((p-1) - 2) + \sqrt{(p-1)^2 - 2(p-1) + 5}.$$

Therefore, the result holds.

ii) Let F be the minimum edge dominating set of graph $\Gamma(\mathbb{Z}_{p^2})$. By appling Lemma 2.10, $|F| = k = \frac{p-1}{2}$. For the lower bound, we apply Lemma 2.2. Therefore, we get

$$\begin{aligned} EE_F(G) &\geq 4\Big(\frac{(p-1)(p-2)}{2} - (p-1)\Big) \\ &= 2(p-1)(p-4). \end{aligned}$$

By applying Lemma 2.8, the edge energy of $\Gamma(\mathbb{Z}_{p^2})$ is equal to $2p^2 - 10p + 8$. Thus using Lemma 2.7, we get

$$\begin{aligned} EE_F(G) &\leq EE(G) + k \\ &= 2p^2 - 10p + 8 + \frac{p-1}{2} \\ &= 2(p^2 - 5p + 4) + \frac{p-1}{2} \\ &= 2(p-1)(p-4) + \frac{p-1}{2} \\ &= \frac{(p-1)(4p-15)}{2}. \end{aligned}$$

□

Theorem 3.3. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph where $n = p^3$ and p a prime. Then,*

i) $LE(\Gamma(\mathbb{Z}_{p^3})) = \frac{(2p-3)(p^3-p^2+1)}{p+1}$.

ii) $LE^+(\Gamma(\mathbb{Z}_{p^3})) = \frac{-p^3+p^2+p+1}{p+1} + \sqrt{p^4 - 6p^2 + 8p + 1}$, *Proof.* Since $\Gamma(\mathbb{Z}_{p^3})$ is a complete split graph of order p^2-1 with independence number p^2-p, thus the number of edges of $\Gamma(\mathbb{Z}_{p^3})$ is equal to

$$m = \frac{(p-1)(p-2)}{2} + p(p-1)^2 = \frac{(p-1)(2p^2-p-2)}{2}.$$

Therefore, in the formulas of the Laplacian and signless Laplacian energies we have, $\frac{2m}{n} = \frac{2p^2-p-2}{p+1}$.

i) Based on the Corollary 10 in [21], the Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^3})$ is $\left\{0, (p-1)^{[p^2-p-1]}, (p^2-1)^{[p-2]}\right\}$. Thus, we get

$$\begin{aligned} LE(G) &= \sum_{i=1}^{p^2-1} |\mu_i - \frac{2m}{n}| \\ &= \left|0 - \frac{2p^2-p-2}{p+1}\right| + (p^2-p-1)\left|(p-1) - \frac{2p^2-p-2}{p+1}\right| \\ &+ (p-2)\left|(p^2-1) - \frac{2p^2-p-2}{p+1}\right| = \frac{(2p-3)(p^3-p^2+1)}{p+1}. \end{aligned}$$

ii) According to the Corollary 3.6 in [19], the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^3})$ is obtained as follows

$$\left\{(p-1)^{[p^2-p-1]}, (p^2-3)^{[p-2]}, \frac{1}{2}\big(p^2-3 \pm \sqrt{\alpha}\big)\right\},$$

where $\alpha = p^4 - 6p^2 + 8p + 1$. Therefore, we have

$$\begin{aligned} LE^+(G) &= \sum_{i=1}^{p^2-1} |\mu_i^+ - \frac{2m}{n}| \\ &= (p^2-p-1)\left|(p-1) - \frac{2p^2-p-2}{p+1}\right| + (p-2)\left|(p^2-3) - \frac{2p^2-p-2}{p+1}\right| \\ &+ \left|\frac{1}{2}\big(p^2-3+\sqrt{\alpha}\big) - \frac{2p^2-p-2}{p+1}\right| + \left|\frac{1}{2}\big(p^2-3-\sqrt{\alpha}\big) - \frac{2p^2-p-2}{p+1}\right| \\ &= \frac{-p^3+p^2+p+1}{p+1} + \sqrt{\alpha}. \end{aligned}$$

With putting $\alpha = p^4 - 6p^2 + 8p + 1$ in the above relation, the result holds. □

Theorem 3.4. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph where $n = p^3$ and $p > 2$ a prime. If F is the minimum edge dominating set of $\Gamma(\mathbb{Z}_{p^3})$, then*

$$2(p-1)\big(\sqrt{1+4p} - 2\big) \leq EE_F\big(\Gamma(\mathbb{Z}_{p^3})\big) \leq \frac{1}{2}\big(8p^2 - 8p - 11\big)(p-1).$$

Proof. According to the definition of the zero-divisor graphs, the graph $\Gamma(\mathbb{Z}_{p^3})$ is the complete split graph with p^2-1 vertices and contains the independent set of cardinality $\alpha = p^2 - p$ and the induced subgraph K_{p-1}. Also, the energy of this graph is $E\big(\Gamma(\mathbb{Z}_{p^3})\big) = (p-1)\sqrt{1+4p}$ [22]. Therefore using Lemma 2.3, we get

$$\begin{aligned} EE_F\big(\Gamma(\mathbb{Z}_{p^3})\big) &\geq 2\big(E\big(\Gamma(\mathbb{Z}_{p^3})\big) - 2n + 2\alpha\big) \\ &= 2\big((p-1\sqrt{1+4p} - 2(p^2-1) + 2(p^2-p)\big) \\ &= 2(p-1)\big(\sqrt{1+4p} - 2\big). \end{aligned}$$

For the upper bound, we suppose F is the minimum edge dominating set of $\Gamma(\mathbb{Z}_{p^3})$. According to the structure complete split graph $\Gamma(\mathbb{Z}_{p^3})$, one can easily show that all of the edges of the minimum edge dominating set in the clique, K_{p-1} dominate other edges in graph $\Gamma(\mathbb{Z}_{p^3})$. Therefore using Lemma 2.10, $|F| = \frac{p-1}{2}$. Therefore, using Lemma 2.1 and since $m = \frac{(p-1)(2p^2-p-2)}{2}$ and $n = p^2 - 1$, we get

$$\begin{aligned} EE_F\big(\Gamma(\mathbb{Z}_{p^3})\big) &\leq 4m - 2n + |F| \\ &= 2(p-1)(2p^2-p-2) - 2(p^2-1) + \frac{p-1}{2}. \end{aligned}$$

The result for the upper bound follows from simplifications of the above relation. □

In the following theorem, we compute the exact value of the minimum dominating energy of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^3})$.

Theorem 3.5. *The minimum dominating energy of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^3})$ for a prime $p > 2$ is equal to $E_D\big(\Gamma(\mathbb{Z}_{p^3})\big) = 1 + (p-1)\sqrt{1+4p}$.*

Proof. Let D be the minimum dominating energy of graph $\Gamma(\mathbb{Z}_{p^3})$ and $\{v_1, v_2, \ldots, v_{p^2-1}\}$ be the vertex set of the graph. By a similar argument as the proof of Theorem 3.4, all of the vertices of the minimum dominating set in the induced subgraph K_{p-1} of $\Gamma(\mathbb{Z}_{p^3})$ dominate other vertices in this graph. Thus, we have $|D| = 1$. Without loss of generality, we suppose $v_1 \in D$. According to the definition of the minimum dominating matrix and the adjacency matrix of $\Gamma(\mathbb{Z}_{p^3})$ in [22], we get the minimum dominating matrix of graph $\Gamma(\mathbb{Z}_{p^3})$ as following

$$A_D\big(\Gamma(\mathbb{Z}_{p^3})\big) = \begin{bmatrix} A'_{p^2-p} & \mathbf{1}_{(p^2-p)\times(p-1)} \\ \mathbf{1}^t_{(p-1)\times(p^2-p)} & \mathbf{1}_{p-1} \end{bmatrix},$$

where $\mathbf{1}$ is the matrix of ones and A' a matrix of order $p^2 - p$ is defined as follows

$$A' = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}.$$

The characteristic polynomial is as follows

$$\big|A_D - \lambda I\big| = (-1)^{p-1}(1-\lambda)\lambda^{p^2-4}\Big(\lambda^2 - (p-1)\lambda - (p^2-p)(p-1)\Big) = 0.$$

Therefore, the spectrum of the minimum dominating matrix $A_D\big(\Gamma(\mathbb{Z}_{p^3})\big)$ is

$$\Big\{1^{[1]}, 0^{[p^2-4]}, \frac{(p-1)(1 \pm \sqrt{1+4p})}{2}\Big\}.$$

Consequently, we obtain the minimum dominating energy of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^3})$ as follows

$$\begin{aligned} E_D\big(\Gamma(\mathbb{Z}_{p^3})\big) &= \sum_{i=1}^{p^2-1} \big|\lambda_i\big| \\ &= 1 + \Big|\frac{(p-1)(1+\sqrt{1+4p})}{2}\Big| + \Big|\frac{(p-1)(1-\sqrt{1+4p})}{2}\Big| \\ &= 1 + (p-1)\sqrt{1+4p}. \end{aligned}$$

□

Now, we consider the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ for $n = pq$ where p and q are primes. In this case, $Z^*(\mathbb{Z}_n) = A \cup B$ in which $A = \{kp \mid k = 1, 2, \ldots, q-1\}$ and $B = \{kq \mid k = 1, 2, \ldots, p-1\}$. One can easily show that for any $x, y \in \mathbb{Z}_n$, $xy = o$ if and only if $x \in A$ and $y \in B$ or $x \in B$ and $y \in A$. Therefore, the zero-divisor graph $\Gamma(\mathbb{Z}_{pq})$ is the complete bipartite graph $K_{p-1,q-1}$. In the following theorems, we obtain energies of graph $\Gamma(\mathbb{Z}_{pq})$ for different cases of p and q.

Theorem 3.6. *Let $\Gamma(\mathbb{Z}_{pq})$ be the zero-divisor graph where p and q are primes. Then,*

i) if $p > q$, then $LE(\Gamma(\mathbb{Z}_{pq})) = LE^+(\Gamma(\mathbb{Z}_{pq})) = \frac{2(q-1)(p^2-pq+2q-2)}{p+q-2}$.

ii) if $p < q$, then $LE(\Gamma(\mathbb{Z}_{pq})) = LE^+(\Gamma(\mathbb{Z}_{pq})) = \frac{2(p-1)(q^2-pq+2p-2)}{p+q-2}$.

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_{pq})$. Since graph $\Gamma(\mathbb{Z}_{pq})$ is a complete bipartite graph, using Lemma 2.6 and [14], the spectrum of Laplacian and signless Laplacian matrices of graph $\Gamma(\mathbb{Z}_{pq})$ is as follows

$$spec\big(L(G)\big) = spec\big(L^+(G)\big) = \big\{(0)^{[1]}, (q-1)^{[p-2]}, (p-1)^{[q-2]}, (p+q-2)^{[1]}\big\}.$$

Since the order and size of graph $\Gamma(\mathbb{Z}_{pq})$ are $p+q-2$ and $(p-1)(q-1)$, respectively, thus using the definition of the Laplacian and signless Laplacian energies, we get

$$\begin{aligned} LE(G) = LE^+(G) &= \sum_{i=1}^{p-1} \big|\mu_i - \frac{2m}{n}\big| \\ &= \Big|0 - \frac{2(p-1)(q-1)}{p+q-2}\Big| + (p-2)\Big|(q-1) - \frac{2(p-1)(q-1)}{p+q-2}\Big| \\ &+ (q-2)\Big|(p-1) - \frac{2(p-1)(q-1)}{p+q-2}\Big| + \Big|(p+q-2) - \frac{2(p-1)(q-1)}{p+q-2}\Big|. \end{aligned}$$

We consider the following cases.

i) If $p > q$, then

$$\begin{aligned} LE(G) = LE^+(G) &= \frac{2(p-1)(q-1)}{p+q-2} - (p-2)\Big((q-1) - \frac{2(p-1)(q-1)}{p+q-2}\Big) \\ &+ (q-2)\Big((p-1) - \frac{2(p-1)(q-1)}{p+q-2}\Big) + \Big((p+q-2) - \frac{2(p-1)(q-1)}{p+q-2}\Big) \\ &= \frac{2(q-1)(p^2-pq+2q-2)}{p+q-2}. \end{aligned}$$

ii) If $p < q$, then

$$\begin{aligned} LE(G) = LE^+(G) &= \frac{2(p-1)(q-1)}{p+q-2} + (p-2)\Big((q-1) - \frac{2(p-1)(q-1)}{p+q-2}\Big) \\ &- (q-2)\Big((p-1) - \frac{2(p-1)(q-1)}{p+q-2}\Big) + \Big((p+q-2) - \frac{2(p-1)(q-1)}{p+q-2}\Big) \\ &= \frac{2(p-1)(q^2-pq+2p-2)}{p+q-2}. \end{aligned}$$

Therefore, the results complete. □

Theorem 3.7. *Let $\Gamma(\mathbb{Z}_{pq})$ be the zero-divisor graph where $p = 2$ and $q > 2$ are primes. If D and F are the minimum dominating and the minimum edge dominating sets of graph $\Gamma(\mathbb{Z}_n)$ and its line graph, respectively such that $|F| = k$, then*

i) $E_D\big(\Gamma(\mathbb{Z}_{pq})\big) = \sqrt{4q-3}$.

ii) $EE_F\big(\Gamma(\mathbb{Z}_{pq})\big) = (q-3) + \sqrt{q^2-4q+8}$.

Proof. The zero-divisor graph $\Gamma(\mathbb{Z}_{2q})$ for the prime number $q > 2$ is a star graph $K_{1,q-1}$. Since $x^{q-2}\big(x^2 - x - (q-1)\big) = 0$ is the characteristic polynomial of the minimum dominating matrix $A_D\big(\Gamma(\mathbb{Z}_{pq})\big)$ [20], then $E_D\big(\Gamma(\mathbb{Z}_{pq})\big) = 2\Big(\frac{1+\sqrt{4q-3}}{2}\Big) = \sqrt{4q-3}$.
Using Lemma 2.5, the minimum edge dominating energy of zero-divisor graph $\Gamma(\mathbb{Z}_{2q})$ is obtained. □

Theorem 3.8. *Let $\Gamma(\mathbb{Z}_{pq})$ be the zero-divisor graph where $p > 2$ and $q > 2$ are distinct primes. If F is the minimum edge dominating set of graph $\Gamma(\mathbb{Z}_n)$, then*

$$\alpha \leq EE_F(G) \leq (\alpha + 4) + min\{p, q\},$$

where $\alpha = 4pq - 8(p + q) + 16$.

Proof. Assume that G is the zero-divisor graph $\Gamma(\mathbb{Z}_{pq})$ where $p > 2$ and $q > 2$ are distinct primes. Since G is a complete bipartite graph $K_{p-1,q-1}$, thus G has $p + q - 2$ vertices and $(p - 1)(q - 1)$ edges. In this case, one can easily obtain the minimum edge domination number of $\Gamma(\mathbb{Z}_{pq})$ is equal to $min\{p, q\}$. So, using Lemma 2.2 we get

$$\begin{aligned} EE_F(G) &\geq 4\big((p-1)(q-1) - (p+q-2)\big) \\ &= 4pq - 8(p+q) + 12. \end{aligned}$$

For the upper bound, we apply Lemmas 2.9 and 2.7. Consequently, we have

$$\begin{aligned} EE_F(G) &\leq EE(G) + k \\ &= 4pq - 8(p+q) + 16 + min\{p, q\} \end{aligned}$$

With putting $\alpha = 4pq - 8(p + q) + 12$, the result completes. □

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