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A note on the Isaacs-Knutson conjecture

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Abstract

Let G be a finite group. Isaacs and Knutson have conjectured that the inequality $\mathrm{dl}(N) \leq |\mathrm{cd}(G|N)|$ holds for all normal solvable subgroups N of G, where $\mathrm{cd}(G|N) = \{\chi(1)|\chi \in \mathrm{Irr}(G) \text{ and } N \nsubseteq \ker\chi\}$ and $\mathrm{dl}(N)$ is the derived length of N. In this paper, we study some analogies between the properties of some counterexamples(if exist) to this conjecture and the Isaacs-Seitz conjecture.

Keywords: Derived length, irreducible character degrees, Normal subgroups.

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1 Introduction

Let G be a finite group and $\mathrm{cd}(G)$ be the set of irreducible character degrees of G. The Isaacs-Seitz conjecture is a well-known open problem on solvable groups which states that $|\mathrm{cd}(G)|$ is an upper bound for the derived length $\mathrm{dl}(G)$ of G where G is a solvable group. In other words, $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$. This inequality is known as Taketa's inequality.

There are many papers investigating this conjecture, we mention just a few examples out of several. Taketa verified that this inequality holds for M-groups, see Corollary 5.13 of [2]. Moreover, Berger in [1] show that this conjecture is true for groups of odd order.

On the other hand, Isaacs and Knutson have conjectured that the inequality $\mathrm{dl}(N) \leq |\mathrm{cd}(G|N)|$ holds for all normal solvable subgroups N of G, where $\mathrm{cd}(G|N) = \{\chi(1)|\chi \in \mathrm{Irr}(G) \text{ and } N \nsubseteq \ker\chi\}$. This conjecture is solved only in some particular cases as yet. For instance, in [3], it has been shown that if N is a nilpotent normal subgroup of G, then this conjecture holds.

Assume that G is solvable and $N = G'$. If the Isaacs-Knutson conjecture is true, then we can write

$$\begin{aligned}\mathrm{dl}(G) &= \mathrm{dl}(N) + 1\\ &\leq |\mathrm{cd}(G|N)| + 1\\ &= |\mathrm{cd}(G)|,\end{aligned}$$

since $\mathrm{cd}(G|G') = \mathrm{cd}(G) - \{1\}$. Therefore, the Isaacs-Seitz conjecture is true.

In this paper, we study some analogies between the properties of some counterexamples to these conjectures.

[1]speaker

2 Main results

The next results are useful tools for the proof of Theorem 2.3.

Lemma 2.1. *Let N and M be normal solvable subgroups of a finite group G such that $M \leq N$. Then*

1. $\mathrm{cd}(G|M) \subseteq \mathrm{cd}(G|N)$
2. $\mathrm{cd}((G/M)|(N/M)) \subseteq \mathrm{cd}(G|N)$

Lemma 2.2 (Corollry 4.3 of [3])**.** *Let $\mathrm{F} = \mathrm{F}(N)$, where $1 < N \lhd G$ and N is solvable, and let $m = max(\mathrm{cd}(G|N))$. Then $m \notin \mathrm{cd}((G/F)|(N/F))$, which is therefore a proper subset of $\mathrm{cd}(G|N)$.*

In the following theorem, we find some properties of a counterexample to the Isaacs-knutson conjecture of minimal order.

Theorem 2.3. *Assume that a normal solvable subgroup N of G be a counterexample to the Isaacs-Knutson conjecture of minimal order. Then the following situations occur:*

1. $\mathrm{F}(N) \neq N$ *is a non-abelian p-group.*
2. $N^{(n-1)}$ *is the unique minimal normal subgroup of N whenever $n = \mathrm{dl}(N)$.*
3. $\mathrm{cd}((G/M)|(N/M)) = \mathrm{cd}(G|N)$ *for each normal subgroup M of G contained in N.*
4. $\mathrm{dl}(N) = |\mathrm{cd}(G|N)| + 1$
5. *N is not a normal Hall subgroup.*

In the following results, some properties of a counterexample to the Isaacs-Seitz conjecture of minimal order have been stated.

Theorem 2.4 (Proposition 2.7 of [5])**.** *Let $\mathcal{P}$ be a class of finite solvable groups which is closed with respect to taking quotients. Suppose there exists a group in $\mathcal{P}$ for which the Taketa inequality is not true and let G be such a group of smallest possible order. Then the following hold:*

1. $G^{(n-1)}$ *is the unique minimal normal subgroup of G where $n = \mathrm{dl}(G)$.*
2. $\mathrm{cd}(G/G^{(n-1)}) = \mathrm{cd}(G)$.
3. $\mathrm{dl}(G) = |\mathrm{cd}(G)| + 1$
4. $\mathrm{F}(G)$, *the Fitting subgroup of G, is a p-group for some prime p.*

Theorem 2.5 (Theorem 2.2 of [4])**.** *If a finite solvable group G is a counterexample to the Isaacs-Seitz conjecture of minimal order, then G does not contain a non-trivial normal Hall subgroup.*

Motivated by the previous results, we put forward the following conjecture:

Conjecture. Assume that a normal solvable subgroup N of G be a counterexample to the Isaacs-Knutson conjecture of minimal order. Then N is a counterexample to the Isaacs-Seitz conjecture.(It does not satisfy the Taketa's inequality.)

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