2-transitive permutation groups with abelian stabilizers having constant movement

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Abstract

In this paper, we determine the structure of 2-transitive permutation groups with abelian point stabilizers having constant movement.

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1 Introduction

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a natural number. If for a subset Γ of Ω the size $|\Gamma^g - \Gamma|$ is bounded, for $g \in G$, the movement of Γ is defined as

$$\text{move}(\Gamma) := \max_{g} |\Gamma^g - \Gamma|.$$

If $\text{move}(\Gamma) \leq m$ for all $\Gamma \subseteq \Omega$, then G is said to have bounded movement m and the movement of G is defined as the

$$\text{move}(G) := \max_{\Gamma, g} |\Gamma^g - \Gamma|.$$

This notion was introduced in [5]. Similarly, for each $g \in G$, the movement of g is defined as:

$$\text{move}(g) := \max_{\Gamma} |\Gamma^g - \Gamma|.$$

If all non-identity elements of G have the same movement, then we say that G has constant movement (see [2]). Clearly, every permutation group with constant movement m has bounded movement m. By Theorem 1 of [5], if G has bounded movement m, then Ω is finite, and its size is bounded by a function of m.

A permutation group G on Ω is called k-transitive, if for every two ordered k-tuples $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ of points of Ω (with $\alpha_i \neq \alpha_j$ for $i \neq j$) there exists $g \in G$ which takes α_i into β_i: $\alpha_i^g = \beta_i$ $(i = 1, \ldots, k)$. As a well-known result, a 2-transitive group G on Ω is primitive and for any $\alpha \in \Omega$, the stabilizers G_α are maximal subgroups of G which act transitively on the set $\Omega - \{\alpha\}$. A Frobenius group G

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on Ω is a non-regular transitive permutation group in which only the identity element fixes more than one point, i.e., $G_{\alpha\beta} = 1$, for any two points $\alpha, \beta \in \Omega$. It is proved that a Frobenius group G is the semi-direct product $K \rtimes H$, with normal subgroup K. The main result of [4] states that a 2-transitive permutation group G with abelian point stabilizer is isomorphic to the affine group $AGL_1(p)$, for a prime number p. We know that the affine group $AGL_1(p)$ is the Frobenius group $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$.

The goal of this paper is to find the structure of 2-transitive permutation groups G with abelian point stabilizers G_α having constant movement m. Note that for $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the integer part of x.

2 Examples and preliminaries

In this section, we give some preliminary results that will be used in the proof of our main results. First, we present a way to calculate the movement of elements of a permutation group.
Let $1 \neq g \in G$ and suppose that g in its disjoint cycle representation has s (s is a positive integer) nontrivial cycles of lengths $l_1, ..., l_s$, say. We might represent g as

$$g = (a_1 a_2 ... a_{l_1})(b_1 b_2 ... b_{l_2}) ... (z_1 z_2 ... z_{l_s}).$$

Let $\Gamma(g)$ denote a subset of Ω consisting of $\lfloor l_i/2 \rfloor$ points from the i^{th} cycle, for each i, chosen in such a way that $\Gamma(g)^g \cap \Gamma(g) = \emptyset$. For example, we could choose

$$\Gamma(g) = \{a_2, a_4, \ldots, a_{k_1}, b_2, b_4, \ldots, b_{k_2}, ..., z_2, z_4, \ldots, z_{k_s}\},$$

where $k_i = l_i - 1$ if l_i is odd and $k_i = l_i$ if l_i is even. Note that $\Gamma(g)$ is not uniquely determined as it depends on the way each cycle is written. For any set $\Gamma(g)$ of this kind, we say that $\Gamma(g)$ *consists of every second point of every cycle of* g. From the definition of $\Gamma(g)$ we see that

$$|\Gamma(g)^g - \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{s} \lfloor l_i/2 \rfloor.$$

The next lemma shows that this quantity is an upper bound for $|\Gamma^g - \Gamma|$ for an arbitrary subset Γ of Ω.

Lemma 2.1. *(Lemma 2.1 of [3]) Let G be a permutation group on a set Ω and suppose that $\Gamma \subseteq \Omega$. Then for each $g \in G$, $|\Gamma^g - \Gamma| \leq \sum_{i=1}^{s} \lfloor \frac{l_i}{2} \rfloor$, where l_i is the length of the i^{th} cycle of g and s is the number of non-trivial cycles of g in its disjoint cycle representation. This upper bound is attained for $\Gamma = \Gamma(g)$ defined above.*

Remark 2.2. ([1]) Let g be an element of a permutation group G on a set Ω. Assume that the set Ω is the disjoint union of G-invariant sets Ω_1 and Ω_2. Then every subset Γ of Ω is a disjoint union of subsets $\Gamma_i = \Gamma \cap \Omega_i$ for $i = 1, 2$. Let g_i be the permutation on Ω_i induced by g for $i = 1, 2$. Since $|\Gamma^g - \Gamma| = |\Gamma_1^{g_1} - \Gamma_1| + |\Gamma_2^{g_2} - \Gamma_2|$, we have:

$$\text{move}_\Omega(g) = \sum_{i=1}^{2} \max\{|\Gamma_i^{g_i} \setminus \Gamma_i| | \Gamma_i \subseteq \Omega_i\} = \text{move}_{\Omega_1}(g_1) + \text{move}_{\Omega_2}(g_2).$$

In [2], the construction of a transitive permutation action of a group G which is a semi-direct product of its two subgroups is given as below.

Let $K = \langle k \rangle$ be cyclic of order n and $H = \langle h \rangle$ be cyclic of order m and suppose r is an integer such that $r^m \equiv 1 \pmod{n}$. For $i = 1, \ldots, m$, let $(h^i)\theta : K \longrightarrow K$ be defined by $k^{(h^i)\theta} = k^{r^i}$ for k in K. It is straightforward to verify that each $(h^i)\theta$ is an automorphism of K, and that θ is a homomorphism from H to $Aut(K)$. Hence the semi-direct product $G = K \rtimes H$ (with respect to θ) exists and if $K = \langle k \rangle$, then G is given by the relations:

$$k^n = 1, \quad h^m = 1, \quad h^{-1}k.h = k^r, \quad with \quad r^m \equiv 1 \pmod{n}.$$

Here every element of G is uniquely expressible as $k^i h^j$, where $0 \leq i \leq n-1$, $0 \leq j \leq m-1$. The semi-direct products of this type (as a permutation group on a set Ω of size n), provide examples of transitive permutation groups where every non-identity element has constant movement. Note that, if $n = p$, a prime, then by Theorem 3.6.1 of [6], this group G is a subgroup of the Frobenius group $AGL_1(p) = \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$.

Lemma 2.3. *(Lemma 2.1 of [2]) Let G be a semi-direct product $\mathbb{Z}_p \rtimes \mathbb{Z}_{2^a}$ denote a group defined as above of order $p.2^a$ where $2^a|(p-1)$ for some $a \geq 1$. Then G acts transitively on a set Ω of size p and in this action all non-identity elements of G have the same movement equal to $(p-1)/2$.*

3 Main results

In this section, we give our main result as follows.

Theorem 3.1. *Let G be a 2-transitive permutation group on a set Ω with abelian point stabilizers. If G has constant movement m, for a natural number m, then $|\Omega| = p$ and G is isomorphic to the affine group $AGL_1(p) = \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$, for some odd prime p, in which $p-1$ is a power of 2. Also, the movement of G is equal to $m = \frac{p-1}{2}$.*

At the end, we verify the movement of these affine groups for $p = 5, 17$ in the following two examples.

Example 3.2. Consider the affine group $G = AGL_1(5) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$ in its 2-transitive permutation representation. Then $G = \langle k, h \rangle$, where $k := (1\,2\,3\,4\,5)$ and $h := (2\,3\,5\,4)$ and $h^{-1}kh = k^2$. In this action, every element of G has one cycle of length 5 or one cycle of length 4 or two cycles of length 2. So, G has constant movement 2.

Example 3.3. Let $G = AGL_1(17) = \mathbb{Z}_{17} \rtimes \mathbb{Z}_{16}$ be the affine group in its 2-transitive permutation representation. Then $G = \langle k, h \rangle$, where $k := (1 \ldots 17)$ and $h := (2\ 4\ 10\ 11\ 14\ 6\ 16\ 12\ 17\ 15\ 9\ 8\ 5\ 13\ 3\ 7)$ and $h^{-1}kh = k^3$. In this action, every element of G has one cycle of length 17 or one cycle of length 16 or two cycles of length 8 or four cycles of length 4 or eight cycles of length 2. So, G has constant movement 8.

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