14th Iranian International Group Theory Conference

Primitive permutation groups with constant movement

Zeinab Foruzanfar
Imam Khomeini International University - Buin Zahra Higher Education Center of Engineering and Technology, Qazvin, Iran

Mehdi Rezaei[1]
Imam Khomeini International University - Buin Zahra Higher Education Center of Engineering and Technology, Qazvin, Iran

Abstract

In this paper, we classify all primitive permutation groups with constant movement.

Keywords: Primitive, transitive group, cycle, movement
Mathematics Subject Classification [2010]: 20B05

1 Introduction

Let G be a permutation group on a set Ω which has no fixed points in Ω and let m be a natural number. If for a subset Γ of Ω the size $|\Gamma^g - \Gamma|$ is bounded, for $g \in G$, the movement of Γ is defined as

$$\text{move}(\Gamma) := \max_{g} |\Gamma^g - \Gamma|.$$

If $\text{move}(\Gamma) \leq m$ for all $\Gamma \subseteq \Omega$, then G is said to have bounded movement m and the movement of G is defined as the

$$\text{move}(G) := \max_{\Gamma, g} |\Gamma^g - \Gamma|.$$

This definition was introduced in [4]. In a similar way, for each $g \in G$, the movement of g is defined as:

$$\text{move}(g) := \max_{\Gamma} |\Gamma^g - \Gamma|.$$

If all non-identity elements of G have the same movement, then G is said to have constant movement (see [2]). Obviously, every permutation group with constant movement m has bounded movement m. It was proved in Theorem 1 of [4] that if G has bounded movement m, then Ω is finite, and its size is bounded by a function of m.

Let G be a permutation group on a set Ω. We call a subset Δ of Ω a block of G if for each $g \in G$ the image set Δ^g either coincides with Δ or has no point in common with Δ. The whole set Ω, the empty set $\emptyset$ and the sets $\{\alpha\}$ consisting of only one point are always blocks of G. These blocks are named as trivial blocks. A transitive permutation group G on Ω is called primitive if G has only trivial blocks. Let $\alpha \in \Omega$. It is known that a transitive permutation group G on Ω is primitive if and only if the stabilizer G_α is a

[1]speaker

maximal subgroup of G. Another important result is that a transitive group of prime degree is primitive. For further information about primitive groups see [6]. We recall that $K \rtimes H$ is a semi-direct product of K and H with normal subgroup K. Also, the dihedral group of order $2n$ is denoted by D_{2n} and for $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the integer part of x.

The goal of this paper is to find the structure of primitive permutation groups G with constant movement m.

2 Examples and preliminaries

In this section, we give some preliminary results that will be used in the proof of our main results. First, we present a way to calculate the movement of elements of a permutation group.
Let $1 \neq g \in G$ and suppose that g in its disjoint cycle representation has s (s is a positive integer) nontrivial cycles of lengths $l_1, ..., l_s$, say. We might represent g as

$$g = (a_1 a_2 ... a_{l_1})(b_1 b_2 ... b_{l_2}) ... (z_1 z_2 ... z_{l_s}).$$

Let $\Gamma(g)$ denote a subset of Ω consisting of $\lfloor l_i/2 \rfloor$ points from the i^{th} cycle, for each i, chosen in such a way that $\Gamma(g)^g \cap \Gamma(g) = \emptyset$. For example, we could choose

$$\Gamma(g) = \{a_2, a_4, \ldots, a_{k_1}, b_2, b_4, \ldots, b_{k_2}, ..., z_2, z_4, \ldots, z_{k_s}\},$$

where $k_i = l_i - 1$ if l_i is odd and $k_i = l_i$ if l_i is even. Note that $\Gamma(g)$ is not uniquely determined as it depends on the way each cycle is written. For any set $\Gamma(g)$ of this kind, we say that $\Gamma(g)$ *consists of every second point of every cycle of* g. From the definition of $\Gamma(g)$ we see that

$$|\Gamma(g)^g - \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{s} \lfloor l_i/2 \rfloor.$$

The next lemma shows that this quantity is an upper bound for $|\Gamma^g - \Gamma|$ for an arbitrary subset Γ of Ω.

Lemma 2.1. *(Lemma 2.1 of [3]) Let G be a permutation group on a set Ω and suppose that $\Gamma \subseteq \Omega$. Then for each $g \in G$, $|\Gamma^g - \Gamma| \leq \sum_{i=1}^{s} \lfloor \frac{l_i}{2} \rfloor$, where l_i is the length of the i^{th} cycle of g and s is the number of non-trivial cycles of g in its disjoint cycle representation. This upper bound is attained for $\Gamma = \Gamma(g)$ defined above.*

The following remark gives us a useful technique for calculating the movement of permutation groups.

Remark 2.2. ([1]) Let g be an element of a permutation group G on a set Ω. Assume that the set Ω is the disjoint union of G-invariant sets Ω_1 and Ω_2. Then every subset Γ of Ω is a disjoint union of subsets $\Gamma_i = \Gamma \cap \Omega_i$ for $i = 1, 2$. Let g_i be the permutation on Ω_i induced by g for $i = 1, 2$. Since $|\Gamma^g - \Gamma| = |\Gamma_1^{g_1} - \Gamma_1| + |\Gamma_2^{g_2} - \Gamma_2|$, we have:

$$\text{move}_{\Omega}(g) = \sum_{i=1}^{2} \max\{|\Gamma_i^{g_i} \setminus \Gamma_i| | \Gamma_i \subseteq \Omega_i\} = \text{move}_{\Omega_1}(g_1) + \text{move}_{\Omega_2}(g_2).$$

Theorem 2.3. *(Theorem 8.8 of [6]) Every normal subgroup $\neq 1$ of a primitive group is transitive.*

Lemma 2.4. *(lemma 1.1 of [2]) Let G be a transitive permutation group on a set Ω such that G has movement m. Then,* **(a)** *if G is a 2-group then $|\Omega| \leq 2m$,* **(b)** *if G is not a 2-group and p is the least odd prime dividing $|G|$, then $|\Omega| \leq \lfloor 2mp/(p-1) \rfloor$.*

Theorem 2.5. *(Theorem 1.1 of [2]) Let m be a positive integer, and let G be a transitive permutation group on a set Ω which has constant movement equal to m. Then G has the maximum possible degree as described in Lemma 2.4, and G is either a p-group in its regular representation, where p is a prime or one of the following holds, when p is the least odd prime dividing the order of G:*
1. *$|\Omega| = p$, $m = (p-1)/2$ and G is the semi-direct product $\mathbb{Z}_p \rtimes \mathbb{Z}_{2^a}$ where $2^a \mid (p-1)$ for some $a \geq 1$;*

2. *$G := A_4$, A_5, $|\Omega| = 6$, and $m = 2$.*
3. *$|\Omega| = 2^s p$ where p is a Mersenne prime, $m = 2^{s-1}(p-1)$, and $1 < 2^s < p$, and G is the semi-direct product $K \rtimes P$ with K a 2-group and $P = \mathbb{Z}_p$ is fixed point free on Ω; K has p-orbits of length 2^s, and each non-identity element of K moves exactly $2^s(p-1)$ points of Ω.*
Moreover, all permutation groups listed above have constant movement.

In the following we give a transitive action of a semi-direct product of two groups from [2].

Let $K = \langle k \rangle$ be cyclic of order n and $H = \langle h \rangle$ be cyclic of order m and suppose r is an integer such that $r^m \equiv 1 \pmod n$. For $i = 1, \ldots, m$, let $(h^i)\theta : K \longrightarrow K$ be defined by $k^{(h^i)\theta} = k^{r^i}$ for k in K. One can see that each $(h^i)\theta$ is an automorphism of K, and that θ is a homomorphism from H to $Aut(K)$. Hence the semidirect product $G = K \rtimes H$ (with respect to θ) exists and if $K = \langle k \rangle$, then G is given by the relations:

$$k^n = 1, \quad h^m = 1, \quad h^{-1}k.h = k^r, \quad with \quad r^m \equiv 1 \pmod n.$$

Here every element of G is uniquely expressible as $k^i h^j$, where $0 \leq i \leq n-1$, $0 \leq j \leq m-1$. The semi-direct products of this type (as a permutation group on a set Ω of size n), provide examples of transitive permutation groups where every non-identity element has constant movement. We note that, if $n = p$, a prime, then by Theorem 3.6.1 of [5], this group G is a subgroup of the Frobenius group $AGL_1(p) = \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$.

Lemma 2.6. *(Lemma 2.1 of [2]) Let G be a semi-direct product $\mathbb{Z}_p \rtimes \mathbb{Z}_{2^a}$ denote a group defined as above of order $p.2^a$ where $2^a|(p-1)$ for some $a \geq 1$. Then G acts transitively on a set Ω of size p and in this action all non-identity elements of G have the same movement equal to $(p-1)/2$.*

Note that since the above action is transitive on the set Ω with $|\Omega| = p$, it is primitive.
Now we give two other examples of primitive groups which has constant movement.

Example 2.7. Let $G = \langle a, b \rangle$, where $a := (2\,3\,4\,5\,6)$ and $b := (1\,3\,5)(2\,6\,4)$. Then $G \cong A_5$ and in this action, every element of G has one cycle of length 5 or two cycles of length 2 or two cycles of length 3. So, G has constant movement 2.

Example 2.8. Let p be a prime and consider the cyclic group $G = \mathbb{Z}_p$ in its regular action on p points. Then every element of G has one cycle of length p. So, G has constant movement $\frac{p-1}{2}$.

3 Main results

In this section, we present our main result as below.

Theorem 3.1. *Let G be a primitive permutation group on a set Ω with no fixed points in Ω. If G has constant movement m, for a natural number m, then:*
(i) $|\Omega| = p$, $m = \frac{p-1}{2}$ and $G = \mathbb{Z}_p$ is the cyclic group of order p, for some prime number p.
(ii) $|\Omega| = p$, $m = \frac{p-1}{2}$ and $G = \mathbb{Z}_p \rtimes \mathbb{Z}_{2^a}$ where $2^a \mid (p-1)$ for some $a \geq 1$ and some odd prime p.
(iii) $|\Omega| = 6$, $m = 2$ and $G = A_5$.

References

[1] M. Alaeiyan, M. Rezaei, *Intransitive permutation groups with bounded movement having maximum degree*, Math Rep 13(63) (2011), 109-115.

[2] M. Alaeiyan, H.A. Tavallaee, *Permutation groups with the same movement*, Carpathian J Math 25 (2009), 147-156.

[3] A. Hassani, M. Alaeiyan (Khayaty), E.I. Khukhro, C.E. Praeger, *Transitive permutation groups with bounded movement having maximal degree*, J Algebra 214 (1999), 317-337.

[4] C.E. Praeger, *On permutation groups with bounded movement*, J Algebra 144 (1991), 436-442.

[5] T. Tsuzuku, *Finite Groups and Finite Geometries*, Cambridge University Press, 1982.

[6] H. Wielandt, *Finite Permutation Groups*, Academic Press, 1964.

Email: zforouzanfar@gmail.com, z.forozanfar@bzeng.ikiu.ac.ir
Email: mehdrezaei@gmail.com, m.rezaei@bzeng.ikiu.ac.ir