14th Iranian International Group Theory Conference

On the commuting graphs of some finite AC-groups

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Abstract

Let G be a finite group. The commuting graph of a non-abelian group G, denoted by $\Gamma(G)$, is a simple undirected graph whose vertex set is $G - Z(G)$, and two vertices x and y are adjacent if and only if $xy = yx$. A group G is called an AC-group if $C_G(x)$ is abelian for all $x \in G - Z(G)$. In this paper, the Laplacian spectrum, signless Laplacian spectrum and normalized Laplacian spectrum of the commuting graphs of some finite AC-groups are computed.

Keywords: Commuting graph, normalized Laplacian spectrum, signless Laplacian spectrum
Mathematics Subject Classification [2010]: 05C25, 05C50, 20D60

1 Introduction

A graph Γ is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of Γ called the edges. The vertex-set of Γ is denoted by $V(\Gamma)$, while the edge-set is denoted by $E(\Gamma)$. All graphs in this paper are assumed to be simple, finite and undirected. The adjacency matrix $A(\Gamma) = (a_{ij})$ is an n-square (0,1)-matrix defined as $a_{ij} = 1$ if and only if v_i and v_j are adjacent, where $V(\Gamma) = \{v_1, \ldots, v_n\}$. The degree of vertex v_i is denoted by $d_\Gamma(v_i)$ and the degree matrix denoted by $D(\Gamma)$ is defined as $D(\Gamma) = diag(d_\Gamma(v_1), d_\Gamma(v_2), \ldots, d_\Gamma(v_n))$, which is the diagonal matrix of vertex degrees. Then, the Laplacian, signless Laplacian and normalized Laplacian matrices of a graph Γ is defined as $L(\Gamma) = D(\Gamma) - A(\Gamma)$, $Q(\Gamma) = D(\Gamma) + A(\Gamma)$ and $\mathcal{L}(\Gamma) = (\mathcal{L}_{ij})$, respectively, where

$$\mathcal{L}_{ij} = \begin{cases} 1 & i = j \text{ and } d_\Gamma(v_i) \neq 0, \\ -\frac{1}{\sqrt{d_\Gamma(v_i) d_\Gamma(v_j)}} & v_i v_j \in E(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a finite non-abelian group with center $Z(G)$. The commuting graph of group G, denoted by $\Gamma(G)$, is a simple undirected graph whose vertex set is $G - Z(G)$, and two vertices x and y are adjacent if and only if $xy = yx$. The commuting graph of finite groups was introduced in the seminal paper of Brauer and Fowler [2] in relationship with the classification of finite simple groups. After that a lot of works have been done on this subject. Recall that the Laplacian spectrum, signless Laplacian spectrum and normalized Laplacian spectrum of graph $\Gamma(G)$ denoted by $Spec_L(\Gamma(G)) = \{\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_l^{a_l}\}$, $Spec_Q(\Gamma(G)) = \{\mu_1^{b_1}, \mu_2^{b_2}, \ldots, \mu_m^{b_m}\}$ and $Spec_{\mathcal{L}}(\Gamma(G)) = \{\gamma_1^{c_1}, \gamma_2^{c_2}, \ldots, \gamma_n^{c_n}\}$, respectively, where $\lambda_1, \lambda_2, \ldots, \lambda_l$ are the eigenvalues of $L(\Gamma(G))$ with multiplicities $a_1, a_2, \ldots, a_l$; $\mu_1, \mu_2, \ldots, \mu_m$ are the eigenvalues of $Q(\Gamma(G))$ with multiplicities $b_1, b_2, \ldots, b_m$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the eigenvalues of $\mathcal{L}(\Gamma(G))$ with multiplicities $c_1, c_2, \ldots, c_n$. A group G is called an AC-group if $C_G(x)$ is abelian for all $x \in G - Z(G)$. The purpose of this paper is to compute the Laplacian spectrum, signless Laplacian spectrum and normalized Laplacian spectrum of the commuting graphs of some finite AC-groups. Note that throughout the paper, G is a finite group.

2 Preliminaries

In this section, we give some preliminary results that will be used in the proof of our main results.

Theorem 2.1. *(Satz 5.9 of [4]) Let G be a finite non-solvable group. Then G is an AC-group if and only if G satisfies one of the following conditions:*
(1) $\frac{G}{Z(G)} \cong PSL(2,p^n)$ *or* $PGL(2,p^n)$ *and* $G' \cong SL(2,p^n)$*, where p is a prime and* $p^n > 3$.
(2) $\frac{G}{Z(G)} \cong PSL(2,9)$ *or* $PGL(2,9)$ *and* $G' \cong PSL(2,9)^*$*, where* $PSL(2,9)^* \cong \langle c_1, c_2, c_3, c_4, k | c_1^3 = c_2^2 = c_3^2 = c_4^2 = (c_1c_2)^3 = (c_1c_3)^2 = (c_2c_3)^3 = (c_3c_4)^3 = k^3, (c_1c_4)^2 = k, c_2c_4 = k^3c_4c_2, kc_i = c_ik (i = 1, \ldots, 4), k^6 = 1\rangle$.

Lemma 2.2. *(Lemma 3.9 of [1]) In the above Theorem, if $p = 2$ or $|Z(G)|$ is odd or $G' \cap Z(G) = 1$, then $G \cong Z(G) \times PSL(2, 2^n)$.*

Proposition 2.3. *(Proposition 3.21 of [1]) Let $G = PSL(2,q)$, where q is a p-power (p prime) and let $k = gcd(q-1, 2)$. Then*
(1) *a Sylow p-subgroup P of G is an elementary abelian group of order q and the number of Sylow p-subgroups of G is $q+1$.*
(2) *G contains a cyclic subgroup A of order $t = \frac{q-1}{k}$ such that $N_G(\langle u \rangle)$ is a dihedral group of order $2t$ for every non-trivial element $u \in A$.*
(3) *G contains a cyclic subgroup B of order $s = \frac{q+1}{k}$ such that $N_G(\langle u \rangle)$ is a dihedral group of order $2s$ for every non-trivial element $u \in B$.*
(4) *The set $\{P^x, A^x, B^x | x \in G\}$ is a partition of G. Suppose a is a non-trivial element of G.*
(5) *If $q > 5$ and $q \equiv 1$ mod 4, then*

$$C_G(a) = \begin{cases} N_G(\langle a \rangle) & \text{if } a^2 = 1 \text{ and } a \in A^x \text{ for some } x \in G, \\ A^x & \text{if } a^2 \neq 1 \text{ and } a \in A^x \text{ for some } x \in G, \\ B^x & \text{if } a \in B^x \text{ for some } x \in G, \\ P^x & \text{if } a \in P^x \text{ for some } x \in G. \end{cases}$$

(6) *If $q > 5$ and $q \equiv 3$ mod 4, then*

$$C_G(a) = \begin{cases} N_G(\langle a \rangle) & \text{if } a^2 = 1 \text{ and } a \in B^x \text{ for some } x \in G, \\ B^x & \text{if } a^2 \neq 1 \text{ and } a \in B^x \text{ for some } x \in G, \\ A^x & \text{if } a \in A^x \text{ for some } x \in G, \\ P^x & \text{if } a \in P^x \text{ for some } x \in G. \end{cases}$$

(7) *If $q \equiv 0$ mod 4, then*

$$C_G(a) = \begin{cases} A^x & \text{if } a \in A^x \text{ for some } x \in G, \\ B^x & \text{if } a \in B^x \text{ for some } x \in G, \\ P^x & \text{if } a \in P^x \text{ for some } x \in G. \end{cases}$$

Lemma 2.4. *The group $G = PSL(2, 2^n)$ is an AC-group.*

Proof. Let $G = PSL(2, 2^n)$. Then G is a non-abelian group of order $2^n(2^{2n} - 1)$ with trivial center. By Proposition 2.3, the set of centralizers of non-trivial elements of G is given by $\{A^x, B^x, P^x | x \in G\}$, where P is an elementary abelian 2-subgroup and A, B are cyclic subgroups of G having order 2^n, $2^n - 1$ and $2^n + 1$, respectively. Also, the number of conjugates of P, A and B in G are $2^n + 1$, $2^{n-1}(2^n + 1)$ and $2^{n-1}(2^n - 1)$, respectively. Therefore G is an AC-group. □

Lemma 2.5. *Let G be a non-abelian AC-group and H be an abelian group. Then $G \times H$ is an AC-group.*

Proof. It is easy to see that $Z(G \times H) = Z(G) \times H$ and $C_G(a_1) \times H, C_G(a_2) \times H, \ldots, C_G(a_n) \times H$ are the distinct centralizers of non-central elements of $G \times H$, for $a_i \in G - Z(G)$, where $1 \leq i \leq n$. Therefore if G is an AC-group, then $G \times H$ is also an AC-group. □

Lemma 2.6. *(Lemma 2.1 of [3]) Let G be a finite non-abelian AC-group. Then the commuting graph of G is given by*

$$\Gamma(G) = \cup_{i=1}^{n} K_{|X_i|-|Z(G)|}$$

where $X_1, X_2, \dots, X_n$ are the distinct centralizers of non-central elements of G.

It is well-known that the Laplacian spectrum, signless Laplacian spectrum and normalized Laplacian spectrum of the complete graph K_n on n vertices are given by

$$Spec_L(K_n) = \{0^1, n^{n-1}\}, \quad Spec_Q(K_n) = \{(2n-2)^1, (n-2)^{n-1}\} \quad \text{and} \quad Spec_{\mathcal{L}}(K_n) = \{0^1, (\frac{n}{n-1})^{n-1}\}$$

As a direct consequence of the above results, we have the following theorem:

Theorem 2.7. *Let G be a non-abelian AC-group and $X_1, X_2, \dots, X_n$ are the distinct centralizers of non-central elements of G. Then we have the following cases:*

(i) The Laplacian spectrum of the commuting graph of G is given by

$$\{0^n, (|X_1| - |Z(G)|)^{|X_1|-|Z(G)|-1}, (|X_2| - |Z(G)|)^{|X_2|-|Z(G)|-1}, \dots, (|X_n| - |Z(G)|)^{|X_n|-|Z(G)|-1}\}$$

(ii) The signless Laplacian spectrum of the commuting graph of G is given by

$$\{(2|X_1|-2|Z(G)|-2)^1, \dots, (2|X_n|-2|Z(G)|-2)^1, (|X_1|-|Z(G)|-2)^{|X_1|-|Z(G)|-1}, \dots, (|X_n|-|Z(G)|-2)^{|X_n|-|Z(G)|-1}\}$$

(iii) The normalized Laplacian spectrum of the commuting graph of G is given by

$$\{0^n, (\frac{|X_1| - |Z(G)|}{|X_1| - |Z(G)| - 1})^{|X_1|-|Z(G)|-1}, \dots, (\frac{|X_n| - |Z(G)|}{|X_n| - |Z(G)| - 1})^{|X_n|-|Z(G)|-1}\}$$

3 Main results

Now, we state our main results in the following two theorems.

Theorem 3.1. *Let G be a non-abelian group such that $\frac{G}{Z(G)} \cong PSL(2, 2^n)$. Then the commuting graph of G is given by*

$$(2^n + 1)K_{|Z(G)|(|P^x|-1)} \cup 2^{n-1}(2^n + 1)K_{|Z(G)|(|A^x|-1)} \cup 2^{n-1}(2^n - 1)K_{|Z(G)|(|B^x|-1)}$$

That is, $(2^n + 1)K_{(2^n-1)|Z(G)|} \cup 2^{n-1}(2^n + 1)K_{(2^n-2)|Z(G)|} \cup 2^{n-1}(2^n - 1)K_{2^n|Z(G)|}$.

Theorem 3.2. *Let G be a non-abelian group such that $\frac{G}{Z(G)} \cong PSL(2, 2^n)$. Then we have the following cases:*

(i) The Laplacian spectrum of the commuting graph of G is given by

$$\{0^{2^{2n}+2^n+1}, (2^n-1)|Z(G)|^{(2^n+1)[(2^n-1)|Z(G)|-1]}, (2^n-2)|Z(G)|^{2^{n-1}(2^n+1)[(2^n-2)|Z(G)|-1]}, 2^n|Z(G)|^{2^{n-1}(2^n-1)[2^n|Z(G)|-1]}\}$$

(ii) The signless Laplacian spectrum of the commuting graph of G is given by

$$\{2[(2^n - 1)|Z(G)| - 1]^{2^n+1}, [(2^n - 1)|Z(G)| - 2]^{(2^n+1)[(2^n-1)|Z(G)|-1]}, 2[(2^n - 2)|Z(G)| - 1]^{2^{n-1}(2^n+1)},$$
$$[(2^n - 2)|Z(G)| - 2]^{2^{n-1}(2^n+1)[(2^n-2)|Z(G)|-1]}, 2[2^n|Z(G)| - 1]^{2^{n-1}(2^n-1)}, [2^n|Z(G)| - 2]^{2^{n-1}(2^n-1)[2^n|Z(G)|-1]}\}$$

(iii) The normalized Laplacian spectrum of the commuting graph of G is given by

$$\{0^{2^{2n}+2^n+1}, (\frac{(2^n - 1)|Z(G)|}{(2^n - 1)|Z(G)| - 1})^{(2^n+1)[(2^n-1)|Z(G)|-1]},$$
$$(\frac{(2^n - 2)|Z(G)|}{(2^n - 2)|Z(G)| - 1})^{2^{n-1}(2^n+1)[(2^n-2)|Z(G)|-1]}, (\frac{2^n|Z(G)|}{2^n|Z(G)| - 1})^{2^{n-1}(2^n-1)[2^n|Z(G)|-1]}\}$$

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