Some properties of ν-isologism and product varieties of pair of groups

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Abstract

Let (G, M) be a pair of groups, in which M is a normal subgroup of G. Assume $\mathcal{V}$ is a variety of groups dfined by the set of laws V. We study some more properties of ν-isologism of the class of pair of groups. In fact, it is shown that the subgroups and quotient groups which is invariant under isologism. Moreover, the properties of pair of nilpotent groups which are invariant under iologism will be studied.

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1 Introduction and preliminaries

In 1940, P. Hall [9], introduced the concept of isoclinism between two groups G and H. This concept is an equivalence relation among all groups and it is weaker than isomorphism. Let F_∞ be a free group generated by a countable set $X = \{x_1, x_2, \ldots\}$. Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups defined by the sets of laws V and W, respectively. B.H. Neumann [5], the new varities are introduced. In 1976, C. R. Leedham et al. [3] introduced the notion Baer-invariants, isologism, varietal laws and homology. Later, N. Hekster [8], define a product varieties of groups.

Heidarian et al. [10], extend the notion of υ-isologism for pair of groups. Two groups G and H are isoclinic if and only if there exist isomorphisms $\alpha : \frac{G}{Z(G)} \longrightarrow \frac{H}{Z(H)}$ and $\beta : G' \longrightarrow H'$ such thatβ is induced by α, which are compatible. Hekester in [7], introduced the concept of nilpotent groups of class at most n and arose the concept of n-isoclinism. A.R. Salemkar et al. [2], extended the concept of isoclinism of pairs of groups. Hedarian et al. [11] extend the concept of n-isoclinism to the class of all pairs of groups. Hassanzadeh et al. [6], verify a new notion of nilpotency for pairs of groups. The main goal of this paper is to investigate the properties of subgroups and quotient groups of pair of groups which are invariant under ν-isologism and we extend the notion of product varieties of pair of groups and obtain some properties of the product varieties. Our results are useful for studying product varieties of pair of groups.

For any group G, clearly the central of G is defined as the follows:

$$Z(G) = \{g \in G : [g, x] = 1, \ for \quad all \quad x \in G\}.$$

Also, one may define the upper central series as follows:

$$1 = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \ldots.$$

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Definition 1.1. Let (G,M) be a pair of groups, where M is a normal subgroup of G. If $\mathcal{V}$ is the variety of groups defined by the set of laws V, we define

$$V(G,M)=\left\langle v(g_1,\dots,g_im,\dots,g_r)v(g_1,\dots,g_r)^{-1}|v\in V,m\in M,g_i\in G,1\le i\le r\right\rangle,$$

$$V^*(G,M)=\{m\in M|v(g_1,\dots,g_im,\dots,g_r)=v(g_1,\dots,g_r),\forall v\in V,g_i\in G,1\le i\le r\}.$$

If $\mathcal{V}$ is the variety of abelian groups, then

$$V^*(G,M)=Z(G,M)=\{m\in M:[m,g]=1,for\quad all\quad g\in G\}$$

and

$$V(G,M)=[G,M]=\gamma_2\left(G,M\right)=\langle[m,g]:m\in M,g\in G\rangle.$$

Using the above definition the upper ν-series of M in G as follows:

$$1=V_0^*\left(G,M\right)\trianglelefteq V_1^*\left(G,M\right)\trianglelefteq V_2^*\left(G,M\right)\trianglelefteq\dots$$

which is defined by

$$\frac{V_{n+1}^*\left(G,M\right)}{V_n^*\left(G,M\right)}=V^*\left(\frac{G}{V_n^*(G,M)},\frac{M}{V_n^*(G,M)}\right).$$

Also, we define the lower ν-series of M in G as follows:

$$V_{n+1}\left(G,M\right)=M\ge V_{n-1}\left(G,M\right)\ge\dots\ge V_1\left(G,M\right)=V\left(G,M\right).$$

A pair of (G,M) is called ν-nilpotent of pair of class n, whenever $V_n^*\left(G,M\right)=M$, for some natural number n.

Using the above definition of $\gamma_2\left(G,M\right)=[G,M]$, one can define the lower central series for the pair (G,M) as follows:

$$M=\gamma_1\left(G,M\right)\ge\gamma_2\left(G,M\right)\ge\dots$$

such that $\gamma_{n+1}\left(G,M\right)=[G,\gamma_n\left(G,M\right)]$.

Let (G,N) and (H,M) be pairs of groups. An homomorphism from (G,N) to (H,M) is a homomorphism$f:G\to H$ such that $f(N)\subseteq M$. We say that (G,N) and (H,M) are isomorphic and written by $(G,N)\cong(H,M)$, if f is an isomorphism and $f(N)=M$.

For arbitrary pairs of groups (G,M) and (H,N), let $\alpha:\frac{G}{V^*(G,M)}\longrightarrow\frac{H}{V^*(H,N)}$, $\beta:[G,M]\longrightarrow[H,N]$ and $\alpha_|:\frac{M}{Z(G,M)}\longrightarrow\frac{N}{Z(H,N)}$, be isomorphisms which are compatible such that the following diagram is commutative.

$$\begin{array}{ccc}\frac{G}{V^*(G,M)}\times\frac{M}{V^*(G,M)} & \overset{((\alpha,\alpha))}{\longrightarrow} & \frac{H}{V^*(H,N)}\times\frac{NH}{V^*(H,N)}\\ \downarrow & & \downarrow\\ V(G,M) & \overset{\beta}{\longrightarrow} & V(H,N),\end{array}\tag{1}$$

The pair of isoclinism $((\alpha,\alpha)$ is called an isologism between the pairs of groups (G,M) and (H,N), and denoted by $(G,M)\sim_\nu(H,N)$. In the above diagram if we replace $\alpha_|$ instead of α, and $\beta:[G,M]\longrightarrow[H,N]$ then ν-isologism between two groups coincides with isoclinism between pair of groups, where $\alpha_|$ is the restriction of α on $M/Z(G,M)$. The following proposition and lemma is very useful for further investigations.

Lemma 1.2. *[8] Let (G,M) be a pair of groups, H and N are a subgroup and normal subgroups of G respectively with $N\subseteq M$. Then*

(i) $(H,H\cap M)\sim_\nu(HV^*(G,M),(H\cap M)V^*(G,M))$.

In particular, if $G=HV^*(G,M)$, then $(H,H\cap M)\sim_\nu(G,M)$. Conversely, if $\frac{H}{V^*(M,H\cap M)}$ satisfies the descending chain condition on normal subgroups and $(H,H\cap M)\sim_\nu(G,M)$, then $G=HV^*(G,M)$.

(ii) $(G/N,M/N)\sim_\nu(G/(N\cap V(G,M)),M/(N\cap V(G,M)))$.

In particular, if $N\cap V(G,M)=1$, then $(G,M)\sim_\nu(G/N,M/N)$. Conversly, if $V(G,M)$ satisfies the ascending chain condition on normal subgroups and $(G,M)\sim_\nu(G/N,M/N)$, then $N\cap V(G,M)=1$.

Lemma 1.3. *[2] Let (α,β) be a $\mathcal{V}$-isologism between (G,M) and (H,N).*

(a) If G_1 is a subgroup of G with $V^*(G,M)\subseteq G_1$ and $\alpha(\frac{G_1}{V^*(G,M)})=\frac{H_1}{V^*(H,N)}$ then

$$(G_1,G_1\cap M)\sim_\nu(H_1,H_1\cap N).$$

(b) If M_1 is a normal subgroup of G with $M_1\subseteq V(G,M)$, then

$$(G/M_1,M/M_1)\sim_\nu(H/\beta(M_1),N/\beta(M_1)).$$

Lemma 1.4. *[10] Let (G,M) be a pair of groups, and $N\trianglelefteq G$ with $N\subseteq M$. Then*

(a) $V(G/N,M/N)=\frac{V(G,M)N}{N}$

(b) If $[G,N]\subseteq V^*(G,M)$, then $[V(G,M),N]=1$. Particurly, $[V(G,M),V^*(G,M)]=1$

(c) If $N\cap V(G,M)$=1, then $N\subseteq V^*(G,M)$ and $V^*(\frac{G}{N},\frac{M}{N})=\frac{V^*(G,M)N}{N}$.

(d) $V^*(\frac{G}{N},\frac{M}{N})\geq\frac{V^*(G,M)N}{N}$.

2 some properties of ν-isologism

In this section, We study some more properties of ν-isologism of the class of pair of groups and it is shown that the subgroups and quotient groups which are invariant under isologism. In the following result, we always assume that $H\leq G$ and $N\trianglelefteq G$ with $N\leq M$.

The following proposition describe some properties of ν-isologism of pairs of groups.

Proposition 2.1. *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, (α,β) be a $\nu-$isologism between (G,M) and (H,N), $M_1\trianglelefteq G$, $M_1\subseteq M$ and $N_1\subseteq N\trianglelefteq H$. If $\alpha(M_1/V^*(G,M))=N_1/V^*(H,N)$, then $\beta([M_1V^*(G,M)])=[N_1V^*(H,N)]$.*

Lemma 2.2. *Let (G,M) be a pair of groups and H be a subgroup of G. If $\frac{H}{V^*(M,H\cap M)}$ satisfies the descending chain condition on normal subgroups and $\frac{G}{V^*(G,M)}\cong\frac{H}{V^*(M,H\cap M)}$, then $(H,H\cap M)\sim_\nu(G,M)$.*

Now we state the main theorem of this section.

Theorem 2.3. *Let (G,M) be a pair of groups, H and N are a subgroup and normal subgroup of G, respectively with $N\leq M$. Then*

(a) For each normal subgroup $K\trianglelefteq G$, if $N\cap V(G,M)=1$, then

$$(G,M)\sim_\nu\left(\frac{G}{K\cap N},\frac{M}{K\cap N}\right).$$

(b) If $N\cap V(G,M)=1$, then

$$(H,H\cap N)\sim_\nu\left(\frac{H}{H\cap N},\frac{H\cap N}{H\cap N}=1\right).$$

(c) If $G=HV^*(G,M)$, then

$$\left(\frac{G}{N\cap V(G,M)},\frac{M}{N\cap V(G,M)}\right)\sim_\nu\left(\frac{HN}{N\cap V(G,M)},\frac{(H\cap N)N=N}{N\cap V(G,M)}\right).$$

(d) For each subgroup $H_1\leq G$, if $G=HV^*(G,M)$, then

$$(G,M)\sim_\nu(\langle H,H_1\rangle,\langle H,H_1\rangle\cap M).$$

3 product varieties of pair of groups

In this section, we define and study some properties the extend product varieties of pair of groups. In the following result, we always assume that $H \leq G$ and $N \trianglelefteq G$ with $N \leq M$.

Definition 3.1. Let $\mathcal{V}$ and $\mathcal{U}$ be varieties of groups, then the product varieties $\mathcal{V} * \mathcal{U}$ of pair of groups (G, M) defined by the relation $U(G, M) \subseteq V^*(G, M)$.

Let $\mathcal{V}$ be a variety of groups and $\mathcal{E}$ be the variety of pair of trivial groups, then clearly

$$\mathcal{E} * \mathcal{V} = \mathcal{V} = \mathcal{V} * \mathcal{E}.$$

If $\mathcal{N}_m$ and $\mathcal{N}_n$ are two varieties of nilpotence pair of groups of class at most m and n, then $\mathcal{N}_m * \mathcal{N}_n = \mathcal{N}_{m+n}$. Also $[\mathcal{V}, \mathcal{E}] = \mathcal{V} * \mathcal{A}$, where $\mathcal{A}$ is a variety of abelian pair of groups, as

$$\mathcal{V} * \mathcal{A} = \{M | V(G, M) \subseteq A^*(G, M) = Z(G, M)\} = \{M | [V(G, M), \mathcal{E}(G, M)] = 1\} = [\mathcal{V} * \mathcal{E}(G, M)].$$

Now we state the main results of this section.

Proposition 3.2. *Let $\mathcal{U}$,$\mathcal{U}_1$, $\mathcal{V}$ and $\mathcal{V}_1$ be varieties pair of groups such that $\mathcal{U} \subseteq \mathcal{U}_1$ and $\mathcal{V} \subseteq \mathcal{V}_1$, then the following conditions are held*

(a) $\mathcal{U} * \mathcal{V} \subseteq \mathcal{U}_1 * \mathcal{V}_1$

(b) $\mathcal{A} * \mathcal{U} \subseteq \mathcal{U} * \mathcal{A}$ *where $\mathcal{A}$ is the variety of abelian groups.*

Lemma 3.3. *Let $\mathcal{U}$,$\mathcal{V}$ be varieties of groups and $\mathcal{W} = \mathcal{U} * \mathcal{V}$, then the following conditions are equivalent:*

(a) $\frac{W^*(G,M)}{V^*(G,M)} \subseteq U^*(\frac{G}{V^*(G,M)}, \frac{M}{V^*(G,M)})$;

(b) $V(G, U(G, N) \cap M) \subseteq W(G, N)$.

The equality happens in case (a) if and only if the equality (b) happens.

Theorem 3.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be varieties of groups such that $\mathcal{W} = \mathcal{U} * \mathcal{V}$, then the following conditions for each pair of groups (G, M) hold :*

(a) $V^*(G, M) \subseteq W^*(G, M)$;

(b) $\frac{W^*(G,M)}{V^*(G,M)} \subseteq U^*(\frac{G}{V^*(G,M)}, \frac{M}{V^*(G,M)}) \subseteq W^*(\frac{G}{V^*(G,M)}, \frac{M}{V^*(G,M)})$.

Proposition 3.5. *Let $\mathcal{U}$ and $\mathcal{V}$ be varieties of groups defines by the sets of laws U and V, respectively, and $\mathcal{W} = \mathcal{U} * \mathcal{V}$, $N \trianglelefteq G$ with $N \subseteq M$, then the following conditions are held.*

(a) $N \subseteq W^*(G, M) \Rightarrow U(G, N) \subseteq V^*(G, M)$;

(b) $N \cap V^*(G, M) = 1 \Rightarrow N^*(G, M) \subseteq U^*(G, M)$.

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