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On the commutativity degree of $\mathrm{PSL}(2, q)$

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Abstract

Let G be a finite group. The commutativity degree of G denoted by $P(G)$ is the probability of two group elements that commute. As a matter of fact, if $C = \{(x, y) \in G \times G \mid xy = yx\}$, then $P(G) = \frac{|C|}{|G|^2}$. In this paper, we are going to find the commutativity degree of the projective special linear group $\mathrm{PSL}(2, q)$, where q is a power of a prime p, and $q \equiv 0 \pmod 4$.

Keywords: Commutativity degree, Non-abelian group, Simple group
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1 Introduction

For a finite group G, the statistical results go back to 1965, with the title: **On some problems of a statistical group-theory**, written by P. Erdős and P. Turán(see [5, 6, 7, 8, 9, 10, 11]). In [8], it has been proven, for a finite group G,

$$|C| = |\{(x, y) \in G \times G \mid xy = yx\}| = |G|k(G),$$

where $k(G)$ is the number of conjugacy classes of G. Moreover, if $C_G(x)$ is considered as the centralizer of $x \in G$, then clearly $(x, y) \in C$ if and only if $y \in C_G(x)$. Therefore, we also have $|C| = \sum_{x \in G} |C_G(x)|$. In 1973, Gustafson, in [14], re-prove the results in [8]. He also, in that paper, considered the probability of two group elements that commuted, for the first time, and proved some valuable results. Since then, the investigations as to this probability have been interesting topics to research, and up to now, many results have been published, see [2, 4, 12], for instance.

The probability of two group elements that commute in a finite group G, which is also called commutativity degree of G, is denoted by $P(G)$, is equal to $\frac{|C|}{|G|^2}$. Therefore, by the above mentioned discussion, we also have,

$$P(G) = \frac{\sum_{x \in G} |C_G(x)|}{|G|^2} = \frac{k(G)}{|G|}.$$

Finding the commutativity degree of a finite group is equivalent to finding the number of conjugacy classes of the group. This relates commutativity degree to many areas of group theory.

It is clear that if G is an abelian group, then $P(G) = 1$. Gustafson [14] and MacHale [16] proved independently in 1974 that for a non-abelian finite group G, the commutativity degree $P(G) \leq \frac{5}{8}$. On the other words, they show that no finite group has commutativity degree in the interval $(\frac{5}{8}, 1)$ and that a group G has commutativity degree $P(G) = \frac{5}{8} = \frac{2^2+2-1}{2^3}$ if and only if $\frac{G}{Z(G)} = \mathbb{Z}_2 \times \mathbb{Z}_2$. In fact, we have the following proposition.

[1] speaker

Proposition 1.1. *[14] Let G be a finite non-Abelian group. Then $P(G) \leq \frac{5}{8}$.*

Other upper bounds for commutativity degree in terms of centralizers have been obtained for the dihedral group D_{2n} by Omer et al. in [17]. A classification of the groups for which the commutativity degree is above $\frac{11}{32}$ was given in 1979 by Rusin [18] that shows any group with commutativity degree in the interval $(\frac{1}{2}, \frac{5}{8}]$ has a commutativity degree of $\frac{1}{2}(1+\frac{1}{2^{2n}})$ for some $n \in \mathbb{N}$.

Moreover, in [3], it was proven that the non-abelian simple group A_5, is the only group with the commutativity degree $\frac{1}{12}$.

Proposition 1.2. *[3] If $P(G) > \frac{1}{12}$, then G is solvable. Further, the only simple non- Abelian group with commutativity degree $\frac{1}{12}$ is A_5.*

In this paper, we will find the commutativity degree of the projective special linear group PSL(2,q), where q is a power of a prime p, and $q \equiv 0 \pmod 4$.

Finally, it is worth to mention that all unexplained notations and terminology for groups are standard and we refer the reader to [13], for instance.

2 Main results

In this section, we consider the commutativity degree of the projective special linear group PSL(2,q), where q is a power of a prime p. Therefore, in the following, the structure of these kinds of finite groups would be needed.

Proposition 2.1 ([1, 15])**.** *Let $G = \mathrm{PSL}(2, q)$, where q is a power of a prime p and let $k = gcd(q-1, 2)$. Then*
(i) *a Sylow p-subgroup P of G is an elementary abelian group of order q and the number of Sylow p-subgroups of G is $q+1$.*
(ii) *G contains a cyclic subgroup A of order $t = \frac{q-1}{k}$ such that $N_G(\langle u\rangle)$ is a dihedral group of order $2t$, for every non-trivial element $u \in A$.*
(iii) *G contains a cyclic subgroup B of order $s = \frac{q+1}{k}$ such that $N_G(\langle u\rangle)$ is a dihedral group of order $2s$, for every non-trivial element $u \in B$.*
(iv) *the set $\{P^x, A^x, B^x \mid x \in G\}$ is a partition for G.*
Moreover, assume that a is a non-central element of G, then:
(v) *if $q \equiv 0 \pmod 4$, then:*

$$C_G(a) = \begin{cases} A^x & a \in A^x,\ x \in G \\ B^x & a \in B^x, x \in G \\ P^x & a \in P^x, x \in G \end{cases}$$

Here, we focus on the commutativity degree of some projective special linear groups.

Theorem 2.2. *The commutativity degree of the projective special linear group $\mathrm{PSL}(2, q)$, where q is a power of a prime p, and $q \equiv 0 \pmod 4$, is $\frac{q^2+q-1}{q(q-1)^2(q+1)}$.*

Proof. We summarize the proof in two steps.

- In the first step, by Proposition 2.1, every $C_G(a)$, for $a \in G \setminus Z(G)$, is actually equal to the conjugation of some subgroups A, B or P (introduced in Proposition 2.1). Therefor, by the orbit-stabilizer theorem, the number of $C_G(a)$, for $a \in G$, would be $[G : N_G(a)]$. Hence if we deduce that n_A and n_B be the number of centralizers of G with the orders of $|A|$ and $|B|$, respectively, then Proposition 2.1 and the fact that $|\mathrm{PSL}(2, q)| = q(q^2-1)$ assure us

$$n_A = \frac{|\mathrm{PSL}(2,q)|}{|N_G(A)|} = \frac{q(q^2-1)}{2(q-1)} = \frac{q(q+1)}{2},$$

and similarly,

$$n_B = \frac{q(q-1)}{2}.$$

- In the second step, by some calculations, we have

$$P(G) = \tfrac{q^2+q-1}{q(q-1)^2(q+1)}.$$

$\square$

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