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Commuting automorphisms of certain p-groups

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Abstract

Let G be a group. An automorphism α of a group G is called a commuting automorphism if each element g in G commutes with it's image $\alpha(g)$ under α. Let $\mathcal{A}(G)$ be the set of all commuting automorphisms of G. A group G is said to be an $\mathcal{A}$-group if $\mathcal{A}(G)$ forms a subgroup of Aut(G). In this paper, let G be a p-group of nilpotency class c where p is an odd prime. We give some sufficient conditions on G such that G is an $\mathcal{A}$-group. Also we show that if G is a p-group of maximal class of order p^n, $n \geq 5$, then $Aut(G)$ is an $\mathcal{A}$-group.

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1 Introduction

Let G be a group. By $Aut(G)$ and $Aut_c(G)$ we denote the group of all automorphisms and the group of all central automorphisms of G, respectively. An automorphism α of G is called a commuting automorphism if $g\alpha(g) = \alpha(g)g$ for all $g \in G$. The set of all commuting automorphisms of the group G is denoted by $\mathcal{A}(G)$. The commuting automorphisms were first considered for rings (see [3] and [6]).

The following problem was proposed by I. N. Herstein to the American Mathematical Monthly: If G is a simple non-abelian group, then $A(G) = 1$ [8]. Giving answer to Hersteins problem, Laffey in 1998 [10], proved that $A(G) = 1$ provided G has no non-trivial abelian normal subgroups. Also, Pettet gave a more general statement proving that $A(G) = 1$ if $Z(G) = 1$ and the commutator subgroup of G is equal to G (see [10]).

By $Z_i(G)$, we denote the i-th terms of the upper central series of G. The i-th terms of the lower central series of G is denoted by $\gamma_i(G)$ and $\gamma_2(G)$is denoted also by G'. We denote the Fratitini subgroup of G by $\phi(G)$. We write $[x, y] = x^{-1}y^{-1}xy$ for all $x, y \in G$, also $[x,_n y] = \big[[x,_{n-1} y], y\big]$ for $n \geq 2$. Let $C_G(\alpha) = \{g \in G : \alpha(g) = g\}$. Finally $d(G)$ denotes the minimum number of generators of G.

Throughout, p denotes a fixed prime. The nilpotency class of G is denoted by $c(G)$. If $|G| = p^n$ and $c(G) = c$, then the coclass of G is $cc(G) = n - c$. A group of coclass 1 and coclass 2 are called p-groups of maximal class and of almost maximal class respectively.

In 2002, Deaconescu, Silberberg and Walls, asked the following questions about the set $\mathcal{A}(G)$:

1) Is it true that the set $\mathcal{A}(G)$ is always a subgroup of $Aut(G)$?

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2) What conditions on G imply the equality $\mathcal{A}(G) = Aut_c(G)$?

They showed that even though $\mathcal{A}(G)$ has a number of the properties of a group, but it is not necessarily a subgroup of $Aut(G)$ (see [5]). Also they give some conditions on G such that $\mathcal{A}(G) = Aut_c(G)$. For example they proved that if G satisfies max on subgroups and $\phi(G) \cap Z(G') = 1$, then $\mathcal{A}(G) = Aut_c(G)$. They showed that if the subgroup of right 2-engel elements of G, $R_2(G)$, coincides with Z(G), then $\mathcal{A}(G) = Aut_c(G)$.

Definition 1.1. A group G is called $\mathcal{A}$-group if the set

$$\mathcal{A}(G) = \{\alpha \in Aut(G) : g\alpha(g) = \alpha(g)g \;\; for \; all \; g \in G\}$$

forms a subgroup of $Aut(G)$.

Vosooghpour and Akhavan-Malayeri [13] showed that for a given prime p, minimum order of a non-$\mathcal{A}$ p-group G is p^5. They proved that there exists a non-$\mathcal{A}$ p-group G of order p^n for all $n \geq 5$.

Fouladi and Orfi showed that, if G is a finite AC-group or a p-group of maximal class or a metacyclic p-group, then G is an $\mathcal{A}$-group. Also they proved that if G is a p-group of maximal class of order p^n, $n \geq 4$, then $\mathcal{A}(G) = Aut_c(G)$ (see [7]).

In 2015 Rai proved that a finite p-group G of coclass 2, for an odd prime p, is an $\mathcal{A}$-group (see [12]). Vosooghpour also proved this result. But Vosooghpour's result never got published, except in her Ph.D. thesis in 2014 [14].

Also, we proved that a finite 2-group G of almost maximal class is an $\mathcal{A}$-group. Therefore if p is a prime number and suppose G is a p-group with $cc(G) \leq 2$, then $G \in \mathcal{A}$ (see [1]).

In [2] we proved that in a finite 2-groups of almost maximal class $\mathcal{A}(G) = Aut_c(G)$, except only for the following five groups.

$$\begin{aligned}
G_1 &= \langle x, y, t : x^{2^{n-2}} = t^2 = y^2 = 1,\; x^y = x^{-1+2^{n-4}}t,\; x^t = x^{2^{n-3}+1},\; t^y = t\rangle,\\
G_2 &= \langle x, y, t : x^{2^{n-2}} = t^2 = 1, y^2 = x^{2^{n-3}},\; x^y = x^{-1+2^{n-4}}t,\; x^t = x^{2^{n-3}+1},\; t^y = t\rangle,\\
G_3 &= \langle a, b : a^2 = b^4 = [a, b, a] = [a, b, b, b] = 1\rangle,\\
G_4 &= \langle a, b : a^2 = [a, b, a] = [a, b, b]b^{-4} = 1\rangle,\\
G_5 &= \langle a, b : a^2b^{-4} = [a, b, a] = [a, b, b]b^{-4} = 1\rangle.
\end{aligned}$$

In this paper, we give some sufficient conditions on G such that G or Aut(G) is an $\mathcal{A}$-group.

First, we collect some results on commuting automorphisms, required in the poof of main results.

Lemma 1.2. ([13, Lemma 2.2]) *Let G be a group of nilpotency class 2. If $d(G/Z(G)) = 2$, then G is an $\mathcal{A}$-group.*

Theorem 1.3. ([5, Theorem 1.4]) *If G is a group and if $\alpha \in \mathcal{A}(G)$, then $[G^2, \alpha] \leq Z_2(G)$.*

Lemma 1.4. ([12, Lemma 3.2]) *Let p be an odd prime and G be a finite p-group such that $Z_2(G)$ is abelian. Then G is an $\mathcal{A}$-group.*

Lemma 1.5. ([5, Lemma 2.2, Lemma 2.4 and Lemma 2.6]) *Let G be a group and $\alpha, \beta \in \mathcal{A}(G)$, then*
(i) $\alpha\beta \in \mathcal{A}(G)$ if and only if $[\alpha(x), \beta(x)] = 1$ for all $x \in G$.
(ii) $[G, \alpha] \in R_2(G)$.
(iii) $[G', \alpha] \in Z(G)$.
(iv) $[G', G] \leq C_G(\alpha)$.
(v) $x^{-1}\alpha(x) \in C_G(G')$ for all $x \in G$.

Lemma 1.6. ([5, Lemma 2.1]) *If $\alpha \in \mathcal{A}(G)$ and $x, y \in G$, then $[\alpha(x), y] = [x, \alpha(y)]$.*

Theorem 1.7. ([12, Theorem 3.1]) *Let G be a finite p-group for an odd prime p. If $[Z_2(G), \mathcal{A}(G)] \leq Z(G)$, then G is an $\mathcal{A}$-group.*

Theorem 1.8. ([7, Lemma 3.1 and Theorem 3.4]) *Let G be a p-group of maximal class and order p^n, where $n \geq 4$. Then $\mathcal{A}(G) = Aut_c(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.*

2 Main results

Let p be an odd prime and G be a finite p-group. Rai in [12], proved that if $|Z_2(G)/Z(G)| = p^2$ and $Z(G) = \gamma_k(G)$ for some $k \geq 2$, then G is an $\mathcal{A}$-group. In the following theorem, we extend his result.

Theorem 2.1. *Let G be a p-group of nilpotency class c where p is an odd prime. If $G' \cap Z(G) = \gamma_c(G)$ and $|Z_2(G)/Z(G)| \leq p^2$, then G is an $\mathcal{A}$-group. In particular if G is non-abelian, then $p||Inn(G) \cap \mathcal{A}(G)|$.*

Proof. If G is an abelian group, then $\mathcal{A}(G) = Aut(G)$. Therefore suppose G is non-abelian. Let $c(G) = 2$. Then $Z_2(G) = G$, therefore by Lemma 1.2, G is an $\mathcal{A}$-group. In continue suppose $c = c(G) \geq 3$. Since G is an of odd order, by Theorem 1.3, we have for all $\alpha \in \mathcal{A}(G)$ and for all $x \in G$, $x^{-1}\alpha(x) \in Z_2(G)$. If $Z_2(G)$ is an abelia group, then by Lemma 1.4, G is an $\mathcal{A}$-group. So let us assume $Z_2(G)$ is non-abelian. Therefore $Z_2(G)/Z(G)$ is an elementary abelian group. Let $Z_2(G)/Z(G) = \langle aZ(G)\rangle \times \langle bZ(G)\rangle$. Since $Z_2(G)$ is non-abelian, we have $1 \neq [a, b] \in Z(G)$ is an element of order p. Note that every element of $Z_2(G)$ can be written as $a^i b^j z$ for some $z \in Z(G)$ and $0 \leq i, j < p$. Suppose $a^i b^j z \in C_G(a) \cap Z_2(G)$ for some $z \in Z(G)$ and $0 \leq i, j < p$. We show that $C_G(a) \cap Z_2(G) = \langle a, Z(G)\rangle$. Suppose $a^i b^j z \in C_G(a) \cap Z_2(G)$, therefore $1 = [a^i b^j z, a] = [b, a]^j$. Since $0 \leq j < p$, we have $j = 0$. This implics that $C_G(a) \cap Z_2(G) = \langle a, Z(G)\rangle$. By a similar argument we obtain that $C_G(b) \cap Z_2(G) = \langle b, Z(G)\rangle$. Let $\alpha \in \mathcal{A}(G)$. Since $[\alpha(a), a] = 1 = [\alpha(b), b]$, we have $\alpha(a) = a^i u$ and $\alpha(b) = b^j v$ where $0 \leq i, j < p$ and $u, v \in Z(G)$. By Lemma 1.6, $[\alpha(a), b] = [a, \alpha(b)]$ and so $[a, b]^i = [a, b]^j$. Since $|[a, b]| = p$ and $0 \leq i, j < p$, we have $i = j$. Since $[a, b] \in G' \cap Z(G) = \gamma_c(G)$ and $c(G) \geq 3$, by Lemma 1.5, we have $[\alpha(a), \alpha(b)] = [a, b]$, so $[a, b]^{i^2} = [a, b]$. Therefore $i \equiv_p 1$ or $i \equiv_p -1$. Now suppose $\alpha(a) = a^{-1}u$ and $\alpha(b) = b^{-1}v$. It easily follows that for all $w \in Z_2(G)$, $\alpha(w) = w^{-1}m$ for some $m \in Z(G)$. Now $\gamma_{c-1}(G) \leq Z_2(G)$, so for all $y \in \gamma_{c-1}(G)$, we have $\alpha(y) = y^{-1}z'$ for some $z' \in Z(G)$. Also by Lemma 1.5, $\alpha(y) = yz''$ for some $z'' \in Z(G)$. Therefore $y^2 \in Z(G)$, so $y \in Z(G)$ because G is an odd prime. So $\gamma_{c-1}(G) = \gamma_c(G)$, a contradiction. Therefore $\alpha(a) = au$ and $\alpha(b) = bv$ where $u, v \in Z(G)$. By Theorem 1.7, proof is complete. Since $Z_2(G)/Z(G) \cong Aut_c(G) \cap Inn(G)$ and $Aut_c(G) \cap Inn(G) \leq \mathcal{A}(G) \cap Inn(G)$, if G is non-abelian, then $p||Inn(G) \cap \mathcal{A}(G)|$. □

In particular, we have the following consequences of above theorem.

Corollary 2.2. *For any positive integer n and any odd prime number p, there exists a finite $\mathcal{A}$ p-group in which $cc(G) = n$.*

Proof. Let $G = \langle a, c | a^{p^n} = c^{p^{n+1}} = 1, c^a = c^{p+1}\rangle$. It is clear that $|G| = p^{2n+1}$. We have $c(G) = n + 1$, $Z(G) = \langle c^{p^n}\rangle = \gamma_{n+1}(G)$ and $Z_2(G)/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore G satisfies the conditions of Theorem 2.1. Hence G is an $\mathcal{A}$-group and $cc(G) = n$. □

Corollary 2.3. *A finite p-group of $cc(G) \leq 2$, for an odd prime p is an $\mathcal{A}$-group.*

Now, for next corollary we need the following definition.

Definition 2.4. Let m, n be integers and $3 \leq m \leq n$. The set of all groups G of order p^n and $c(G) = m-1$, in which $|\gamma_i(G) : \gamma_{i+1}(G)| = p$ $(i = 2, ..., m-1)$ is denoted by $CF(m, n, p)$.

In [4] Blackburn studied this class of groups.

Theorem 2.5. ([4, Theorem 2.4]) *If $G \in CF(m, n, p)$, then $Z_i(G) \cap G' = \gamma_{m-i}(G)$ for $0 \leq i \leq m-2$.*

Corollary 2.6. *If $G \in CF(m, n, p)$, p is an odd prime such that $|Z_2(G)/Z(G)| \leq p^2$, then G is an $\mathcal{A}$-group.*

Proof. By Theorem 2.5, $G' \cap Z(G) = \gamma_{m-1}(G)$ and by Theorem 2.1, proof is complete. □

Let G be a group of maximal class of order p^n, $n \geq 4$, Where p is a prime. For each i with $2 \leq i \leq n-2$, the 2-step centralizer K_i in G is defined to be centralizer in G of $\gamma_i(G)/\gamma_{i+2}(G)$. Define $P_i = P_i(G)$ by $P_0(G) = G$, $P_1(G) = K_2$ and $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. Take $s_1 \in P_1 - P_2$, $s \in G - \cup_{i=2}^{n-2} K_i$ and define $s_i = [s_{1,i-1}\, s]$ for $2 \leq i \leq n-1$. Note that $G = \langle s, s_1 \rangle$, $P_i = \langle s_i, ..., s_{n-1} \rangle$ for $1 \leq i \leq n-1$ and $Z_i(G) = \gamma_{n-i}(G)$ for $1 \leq i \leq n-1$ (see [9] and [11]).

Theorem 2.7. *If G is a p-group of maximal class of order p^n, $n \geq 5$, then $Aut(G)$ is an $\mathcal{A}$-group.*

Proof. First we show that $R_2(Aut(G))$ is an abelian group. Let $\alpha \in R_2(Aut(G))$. Then $1 = [\alpha, T_x, T_x] = T_{[x^{-1}, \alpha^{-1}(x)x^{-1}]}$ for all $x \in G$. So $[\alpha(x), x] \in Z(G)$. Let $\overline{\alpha}$ be the automorphism induced by α on $\overline{G} = G/Z(G)$. Then $\overline{\alpha} \in \mathcal{A}(G/Z(G)) = Aut_c(G/Z(G))$ by Theorem 1.8. So $g^{-1}\alpha(g) \in Z_2(G)$ for all $g \in G$. Since G is a p-group of maximal class of order p^n, $n \geq 5$, we have $Z_2(G) = \gamma_{n-2}(G)$ is an abelian group.

Now $G = \langle s, s_1 \rangle$. Let $\alpha(s) = sz$ and $\alpha(s_1) = s_1 z_1$ where $z, z_1 \in Z_2(G)$. We claim that $\gamma_3(G) \leq C_G(\alpha)$. Since $[s_i, s] = s_{i+1}$, by a simple calculation, we have $\alpha(s_2) = [\alpha(s_1), \alpha(s)] = s_2 u$ where $u \in Z(G)$. Also $\alpha(s_3) = [\alpha(s_2), \alpha(s)] = [s_2 u, sz] = [s_2, z][s_2, s]^z = s_3$ because $[G', Z_2(G)] = 1$ and $u \in Z(G)$. Since for all $3 \leq i \leq n-2$, $s_{i+1} = [s_i, s]$, we have $\alpha(s_{i+1}) = [\alpha(s_i), \alpha(s)] = [s_i, sz] = [s_i, z][s_i, s]^z = s_{i+1}$ because $[G', Z_2(G)] = 1$. Therefore $\gamma_3(G) \leq C_G(\alpha)$.

Now let $\theta, \gamma \in R_2(Aut(G))$. Therefore for all $g \in G$, we have $\theta(g) = gm$ and $\gamma(g) = gn$ where $m, n \in Z_2(G)$. Since for all $\alpha \in R_2(Aut(G))$, we have $Z_2(G) = \gamma_{n-2}(G) \leq C_G(\alpha)$ is a group of order p^2, a simple calculation shows that $\theta\gamma = \gamma\theta$. Consequently $R_2(Aut(G))$ is an abelian group.

Let $\alpha, \beta \in \mathcal{A}(Aut(G))$ and $x \in Aut(G)$. Then $\alpha(x) = xz_1$, $\beta(x) = xz_2$ where $z_1, z_2 \in C_{Aut(G)}(Aut(G)') \cap R_2(Aut(G))$ by Lemma 1.5. So

$$[\alpha(x), \beta(x)] = [xz_1, xz_2] = [x, xz_2]^{z_1}[z_1, xz_2] = [x, xz_2][z_1, xz_2],$$

because z_1 commutes with every commutator. Since $[\alpha(x), x] = [\beta(x), x] = 1$, we have $[\alpha(x), \beta(x)] = [z_1, z_2]$. Since z_1, z_2 belong to an abelian subgroup of $Aut(G)$, we have $[\alpha(x), \beta(x)] = 1$. So by Lemma 1.5, $Aut(G)$ is an $\mathcal{A}$-group. □

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