Tit's Alternatine and Skew Linear Groups

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ABSTRACT

Let D be a division ring with center F, when $[D:F]$ is finite. Suppose that N is a subnormal subgroup of D^* and M be a non-abelian maximal subgroup of N, then either M contains a non-cyclic free subgroup or one of the following statements holds:

1. Every normal subgroup of D^* is either central or contains D'.
2. There exists a non-central normal subgroup N of D^* and a maximal subfield K of D such that K^* is normal in M and $Gal(K = F) = C_p$, for some prime p. Furthermore, for every such N, $M / N \cong K^*$ In particular, M is metabelian and $[D : F] = p^2$.

KEYWORDS: Skew Linear Groups, maximal subgroup, Tit's Alternative, Free subgroup.

1 INTRODUCTION

Assume that D be a non-commutative division algebra with center F. The structure of subgroups of the group $D^* = D\setminus\{0\}$, is unknown in general. Finite subgroups of D^* have been investigated by Amistur in [1]. Also, normal and subnormal subgroups of D^* have been classified over the last 70 years. Herstein in [5] showed that the number of conjugates of a non-central element of D is infinite.

The topic of the existence of non-cyclic free subgroups in $GL_n(F)$ over a field F was studied by Tits in [7] which investigated that in the characteristic 0, every subgroup of the $GL_n(F)$ over a field F either contains a noncyclic free subgroup or is soluble-by-finite, and every finitely generated subgroup either contains a non-cyclic free subgroup or is soluble-by-finite in the case of prime characteristic. This result of Tits is now referred as the Tits Alternative.

Recently, in several articles, the same issue has been examined in subgroups. The following results are important results that have been achieved in this regard.

Theorem A. Let D be a division ring with center F. Assume that M is a non-abelian maximal subgroup of $GLn_n(D)$. Then, either M contains a non-cyclic free subgroup or there exists a unique maximal subfield K of $M_n(D)$ such that $N_{GLn(D)}(K *) = M, K^* \lhd M, K/F$ is Galois with $Gal(K/F) \cong M/A$, and $Gal(K/F)$ is a finite simple group and $F[M] = M_n(D)$.

Theorem B. Let D be a division ring with center F. Suppose that N is a non-central normal subgroup of $Gl_n(D)$ with $n \geq 1$. Given a maximal subgroup M of N, then either M contains a non-cyclic free subgroup or there exists an abelian subgroup A and a finite family $\{K_i\}_r$ of fields properly containing F with $K_i \cap N \subseteq M$ for all $1 \leq i \leq r$ such that M/A is finite if $Char\ F = 0$ and M/A is locally finite if $Char\ F = p > 0$, where $A \subseteq K_1 \times \cdots \times K_r$.

Theorem C. Let D be a division ring with center F, when $[D: F]$ is finite. Suppose that N is a subnormal subgroup of $Gl_n(D)$ and M be a non-abelian maximal subgroup of N, then either M contains a non-cyclic free subgroup or there exists a non-central maximal normal abelian subgroup A of M such that $K = F[A]$ is a maximal subfield of $M_n(D)$, K/F is Galois and $Gal(K/F) \cong M/(K^* \cap N)$, also $Gal(K/F)$ is a finite simple group with $F[M] = M_n(D)$.

For more information in this field, you can see references [2, 3, 4].

In fact, the Tit's alternative is a very instructive algorithm for examining group properties in group theory. Examining the existence of a free subgroup allows us to have powerful tools for examining the rest of the group properties of a group. For example, to examine group properties such as solubility or locally finite condition using this alternative is a way forward.

We say that a linear group is not amenable when it contains a non-abelian free group. We know that the Tits alternative is an important tool in the proof of Gromov's theorem on groups of polynomial growth. On the other hand, the alternative essentially establishes the result for linear groups and it reduces it to the case of soluble groups.

Also, in geometric group theory, a group H is said to satisfy the Tits alternative if for every subgroup G of H either G is virtually soluble or G contains a non-abelian free subgroup.

Some examples of groups satisfying the are:

- Hyperbolic groups
- Mapping class groups;
- Out(F_n);
- Certain groups of birational transformations of algebraic surfaces.

- Examples of groups not satisfying the Tits alternative are:

- The Grigorchuk group;
- Thompson's group.

2 MAIN RESULT

MainTheorem. Let D be a division ring with center F, when [D: F] is finite. Suppose that N is a subnormal subgroup of D^* and M be a non-abelian maximal subgroup of N, then either M contains a non-cyclic free subgroup or one of the following statements holds:

1. Every normal subgroup of D^* is either central or contains D'.
2. There exists a non-central normal subgroup N of D^* and a maximal subfield K of D such that K^* is normal in M and $Gal(K = F) = C_p$, for some prime p. Furthermore, for every such N, $M / N \cong K^*$ In particular, M is metabelian and $[D : F] = p^2$.

Proof.

Assume that N be a non-central normal subgroup of D^* such that $D' \nsubseteq N$. So by using the

Cartan-Brauer-Hua Theorem implies that $F[N] = D$ which means that N is non- abelian. Consequently, we may conclude that, either M is abelian or there exists a maximal subfield K of D such that K^* is normal in M and

$$M/K^* = Gal(K/F).$$

First, assume that M is abelian, then $N \nsubseteq M$. Thus,

$$D^*/N = MN/N = M/(M \cap N)$$

is an abelian group, which is a contradiction. Therefore, there exists a maximal subfield K of D when,

$$M/K^* = Gal(K/F) := S,$$

where S is simple group.
Now, we conclude that N contains a non-cyclic free subgroup and hence $N \nsubseteq M$ Therefore, we may consider the following two cases:

a. $N \cap M \nsubseteq K^*$: So we may obtain that

$$D^*/N = MN/N \cong M/(M \cap N) \cong K^*(M \cap N)/(M \cap N) \cong K^*/(N \cap K^*)$$

and hence $D' \subseteq N$, which is impossible.

b. $N \cap M \subseteq K^*$: This implies that

$$\frac{D^*}{N} = \frac{MN}{N} \cong \frac{M}{M \cap N}.$$

Assume that $D^* = K^*N$, thus

$$\frac{D^*}{N} \cong \frac{K^*N}{N} \cong \frac{K^*}{K^* \cap N}.$$

Thus, $D' \subseteq N$ which is a contradiction. Hence, MN/NK^* is a non-trivial homomorphic image of the simple group M/K^* and so

$$\frac{D^*}{NK^*} = \frac{MN}{NK^*} \cong \frac{M}{K^*}.$$

Now by [22], which asserts that the multiplicative group of a finite dimensional division algebra admits no non-abelian simple finite quotient, we conclude that

$$\frac{D^*}{K^*N} \cong \frac{M}{K^*} \cong C_p.$$

Therefore, M is metabelian, and $[D : F] = p^2$.

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