Solubility Condition on Subgroups of $Gl_n(D)$

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ABSTRACT

Consider that D be a given division ring with center F, the structure of the group $Gl_n(D)$ for n ≥ 1, in general, is unknown. In these studies and researches, various theorems and various results have been obtained. A strong result has recently been obtained in major work by Rapinchuk, Segev and Seitz in [5]. In this article, authors proved that a normal subgroup of finite dimensional division ring which has a finite quotient in D^* contains one of the groups appearing in the derived series of D^*. Therefore, the quotient group is solvable.

Let D be a non-commutative division ring, not necessarily non-commutative. Also, consider that M an imprimitive maximal subgroup of $Gl_n(D)$, we try to classify solvable imprimitive maximal subgroup of $Gl_n(D)$.

KEYWORDS: Solvable, Division ring, Skew linear group, Maximal subgroup.

1 INTRODUCTION

Consider that D be a given division ring with center F, the structure of the group $Gl_n(D)$ for n ≥ 1, in general, is unknown. In these studies and researches, various theorems and various results have been obtained. Dear readers, for more information and familiarity with classical concepts in this field, you refer to two important references [6] and [7]. A good account of the most important results concerning skew linear groups can be found in [6], as well as in [7] particularly for soluble linear groups. Assume that $n = 1$, the question of the existence of maximal subgroups in D^* has not been completely answered yet. But, for example we know that that $\mathbb{C}^* \cup \mathbb{C}^* j$ is a soluble maximal subgroup of the multiplicative group of the real quaternion division algebra $\mathbb{H}$.

Stuth proved that if an element commutes with a non-central subnormal subgroup of a division ring, then it is central. Consequently, he proved that if $[x, H] \subseteq F$ where H is a subnormal subgroup of multiplicative subgroup of division ring, and $[x, H] = \{xhx^{-1}h^{-1} \mid h \in H\}$ then x is in F. He obtained that a subnormal subgroup of a division ring could not be soluble. Also, a strong result has recently been obtained in major work by Rapinchuk, Segev and Seitz in [5]. In this article, authors proved that a normal subgroup of finite dimensional division ring which has a finite quotient in D^* contains one of the groups appearing in the derived series of D^*. Therefore, the quotient group is solvable. Also in this manner, it is shown that there is a close connection between the question of the existence of maximal subgroups in the multiplicative group of a finite-dimensional division ring and the faomus Albert's conjecture concerning the cyclicity of division algebras of prime degree.

In addition, we know that the monomial subgroup of $Gl_n(D)$ is maximal. The structure of nilpotent maximal subgroups of $Gl_n(D)$ is studied in [1], and it is shown that the nilpotent maximal subgroups of

$Gl_n(D)$are abelian. In addition in this direction, the authors in [1] give a classification of all reducible maximal subgroups of $Gl_n(D)$, and they conjecture that $Gl_n(D)$ contains no soluble maximal subgroups if $n > 1$. Now, we know that this conjecture is true for non-abelian soluble maximal subgroups. We also know the case in which $D = F$ is commutative and hence turn to another conjecture of [1] which asserts that if F is a field and $n \geq 5$, then $Gl_n(F)$ contains no soluble maximal subgroups. This conjecture answered positively for algebraically closed and real closed fields. Indeed, a description of all the soluble maximal subgroups of $Gl_n(F)$ for algebraically closed and real closed fields F is presented. For a field F with Char F ≥ 2, the soluble maximal subgroups of $Gl_2(F)$ are completely determined.

The following theorems have recently been proved in this regard. For further reading in this field, we refer the esteemed readers to references [1, 2, 3 and 4].

Theorem A. Let D be a non-commutative division ring, M a maximal subgroup of $Gl_n(D)$, $n \geq 2$, and H a normal subgroup of M such that M=H is locally finite.

- If M is absolutely irreducible, then H is locally soluble (or FC-group) iff H is abelian.
- If M is not absolutely irreducible and char $D = 0$, then H is locally soluble (or FC-group) iff M is abelian.
- If the center of D contains at least five elements, then H is soluble iff H is abelian.

Theorem B. Given an F-central division algebra D and N a subnormal subgroup of $Gl_n(D)$, if M is a non-abelian absolutely irreducible soluble maximal subgroup of N, then, $n = 1$ and there exists a non-central maximal normal abelian subgroup A of M such that $K = F[A]$ is a maximal subfield of D. Also, D is cyclic of prime degree p such that the groups $Gal\left(\frac{K}{F}\right)$ and $\frac{M}{K^* \cap N}$ are isomorphic. Furthermore, for any $x \in M \backslash K^*$, we have $x^p \in F^*$and $D = F[M] = \bigoplus_{i=1}^{p} Kx^i$.

There is also a very nice theorem from reference [6] in this field.

Theorem C. Let D be a division ring of characteristic zero, n a natural number, and G a locally finite subgroup of $Gl_n(D)$. If G is locally solvable, then G has a metabelian normal subgroup of finite index. In particular, G is solvable.

2 MAIN RESULT

Main Result. Let D be a non-commutative division ring, not necessarily non-commutative. Also, consider that M an imprimitive maximal subgroup of $Gl_n(D)$, we try to classify solvable imprimitive maximal subgroup of $Gl_n(D)$.

Proof. Consider that G is an irreducible group, we say that G is imprimitive if for some natural number $m \geq 2$, there exist some subspaces $V_1, \cdots, V_m$ of D^n when $V = \bigoplus_{i=1}^{m} V_i$. Also, we have for any $g \in G$ the permutation $V_i \rightarrow gV_i$ on the set $\{V_1, \cdots, V_m\}$; otherwise G is primitive.

In addition, we define here the wreath product of a skew linear group and a symetric group. Assume that U be a linear space over a division algebra D, also G_1 be a subgroup of $GL(U)$, Γ a subgroup of a permutation group S_k on $\{1, \ldots, k\}, k > 1$. We define the Cartesian product $U_k = V_1$ as a linear space over D, and we choose any element $v \in V_1$ as $v = (u_1, \ldots, u_k), u_j \in U$. For $f_1, \ldots, f_k \in G_1$, $s \in \Gamma$, we have a mapping $f : V_1 \to V_1, f = \langle f_1, \ldots, f_k, s \rangle$, by setting

$$f(v) = f(u_1, \ldots, u_k) = \bar{v}$$

where the $s(v)$th component of $\bar{v}$ is $f_v(u_v)$, $v = 1, \ldots, k$. In this manner f is an automorphism of V_1.

Consequently, the group of all such functions is called the wreath product of the group G_1 and the symmetric group Γ, and is denoted by $G_1 \wr \Gamma$. Hence, the group $G_1 \wr \Gamma$ is an imprimitive group.

In addition, by Lemma 5 of [7, p. 108], we obtain that any imprimitive subgroup P of $GL_n(D)$ is conjugate to a subgroup of

$$(GL_r(D) \wr Sk), n = rk$$

for some integers r and k with $k > 1$.

Notice that a monomial matrix is a square matrix with exactly one non-zero entry in each row and column. It is easily checked that the set of all n × n monomial matrices over D is conjugate to $D \wr S_n$, when $n > 1$.

Set $n = rk$, for some integers r and k, when $k > 1$. Thus,

$$(GL_r(D) \wr S_k) \subseteq Gl_n(D).$$

Consider that A be the set of all $k \times k$ monomial matrices with entries in D and choose B in A. So, we have a new matrix in $GL_n(D)$ as follows. We replace each nonzero entry in A with a matrix from $Gl_r(D)$. In addition, we replace each zero entry in A with the zero matrix from $M_r(D)$. Let the set of these new matrices by C. Hence $B \subseteq GL_n(D)$. It is easily seen that

$$C = (GL_r(D) \wr S_k).$$

Now, assume that M be an imprimitive maximal solvable subgroup of $GL_n(D)$. As above, M is conjugate to $GL_r(D) \wr S_k$, when $n = rk$. But M contains a copy $GL_r(D)$ and S_k.

First assume that F is an infinite field. Therefore, M is conjugate to $D \wr S_n$, when $D = F$ is commutative and $n \leq 4$. But, since for $n \leq 4$, S_n is a solvable group, we conclude that $F^* \wr S_n$ is an imprimitive maximal solvable subgroup of $GL_n(F)$, for $n \leq 4$.

Now, assume that $D = F$ and F is a finite field. It is easily checked that $GL_2(F_2) \wr S_r$ and $GL_2(F_3) \wr S_r$, for $r \leq 4$, are imprimitive solvable subgroup of $GL_{2r}(F_2)$ and $GL_{2r}(F_3)$.

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