Locally Finite Skew Linear Groups

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ABSTRACT

In the group theory, a locally finite group is a group that can be studied in ways analogous to a finite group. Sylow subgroups, Carter subgroups, and abelian subgroups of locally finite groups have been studied yet

Let D be a division ring with centre F and N a subnormal subgroup of $Gl_n(D)$ and M a maximal subgroup of N. Suppose $D \neq F$ or $n > 1$. If M' is locally finite, then either is M' abelian or F is locally finite.

KEYWORDS: Skew linear group, Locally finite, Maximal subgroup, Division ring.

1 INTRODUCTION

Assume that D be a division ring with center F and let G be a subgroup of $Gl_n(D)$. Set $F[G]$ the F-linear hull of G, i.e., the F-algebra generated by elements of G over F. We also define D^n the space of row n-vectors over D. Thus D^n is a $D-G$-bimodule in the classic manner. G is said to be an irreducible (reducible) subgroup of $Gl_n(D)$ whenever D^n is irreducible (reducible) as a $D-G$-bimodule. Now assume that the elements of D^n as column vectors, we may regard D^n as a $G-D$- bimodule precisely when it has the property as a $D-G$-bimodule. We also say that G is absolutely irreducible if $M_n(D) = F[G]$. For any group G, $Z(G)$ is its centre. For a subgroup H of G, $N_G(H)$ means the normalizer of H in G and $[G : H]$ denotes the index of H in G, and $< H, K >$ is thegroup generated by H and K, when K is a subgroup of G. Assume that S be a subset of $M_n(D)$. Then the centralizer of S in $M_n(D)$ is also defined by $C_{M_n(D)}(S)$. Also denote centre FI of $M_n(D)$ with F. In this paper, $G^{(r)}$ is stated as the r-th derived group of . For a ring R, the group of units of R is denoted by R^*.

Let $K[x_1, \dots, x_n]$ be the polynomial ring over K in the non-commuting indeterminates $x_1; \dots ; x_n$. An algebra P over K is said to satisfy a polynomial identity (or PI) if there exists $f(x_1, \dots, x_n)$ in $K[x_1, \dots, x_n]$ with $f \neq 0$ such that $f(a_1, \dots, a_n)$ for all $a_1, \dots, a_n$ in P.

In the group theory, a locally finite group is a group that can be studied in ways analogous to a finite group. Sylow subgroups, Carter subgroups, and abelian subgroups of locally finite groups have been studied yet. In addition, we know that the topic is credited to work in the 1930s by Russian mathematician Sergei Chernikov. Also, a locally finite groups satisfies a weaker form of Sylow's theorems.

When a locally finite group has a finite p-subgroup contained in no other p-subgroups, then all maximal p-subgroups are conjugate and finite. We know that when there are finitely many conjugates, thus the number of conjugates is congruent to 1 modulo p. Also, if every countable subgroup of a locally finite group has only countably many maximal p-subgroups, then every maximal p-subgroup of the group is conjugate similarly to the Burnside problem. When this need not be true in general, a result of Philip Hall and others is that every infinite locally finite group contains an infinite abelian group. In fact, the proof of this theorem in infinite group theory has been established on the Feit–Thompson theorem on the solubility of finite groups of odd order.

In these studies, it has been done on skew linear groups. Below we mention some of them. For further reading, you can see reference [1] and [3-6]. See also References [2] and [7] for more information on the properties of skew linear groups.

Theorem A. Let D be a division ring with center F and M be a maximal subgroup of $Gl_n(D)$, then $M = (M / F^*)$ cannot be locally finite unless $char\ F = p > 0$ and either:

(a) $[D : F] = p^2, n = 1$, and $M \cup \{0\}$ is a maximal subfield of D, or
(b) $D = F, n = p$, and $M \cup \{0\}$ is a maximal subfield of $M_p(F)$, or
(c) $D = F$ and F is locally finite.

Theorem B. Let D be a non-commutative division ring, n a natural number, N a subnormal subgroup of $Gl_n(D)$ and M a maximal subgroup of N. If M is locally finite over F, then M is either absolutely reducible or abelian. Moreover, if $n > 1$ and M is not absolutely irreducible, then $[D : F] < 1$.

Theorem C. Let D be a division algebra of finite dimension over its centre F. Suppose that M is a maximal subgroup of $D^* \neq F^*$ and $M/(M \cap F^*)$ is torsion, then $F^* \subset M, M = K^*$ for a maximal subfield K of D, F has characteristic $p > 0, K = F$ is purely inseparable, and D has degree p.

2 MAIN RESULT

Main Theorem. Let D be a division ring with centre F and N a subnormal subgroup of $Gl_n(D)$ and M a maximal subgroup of N. Suppose $D \neq F$ or $n > 1$. If M' is locally finite, then either is M' abelian or F is locally finite.

Proof.

Set $L = F[M] \cap N$. By maximality of M in N, we have either $L = N$or $L = M$. In the first case, we obtain that $N \subseteq F[M]$. So, $SL_n(D) \subseteq N$and thus N is a normal subgroup of $GL_n(D)$. Thus, $SL_n(D) \subseteq F[M]^*$. Therefore, we have $F[M] = M_n(D)$. In the second case, M is a normal subgroup of $F[M]^*$. Assume that M is imprimitive. Thus, M contains an isomorphic copy of D', which is a contradiction. Consider that M is primitive, then $F[M]$ is a prime ring by Goldie Theorem. The $Z(F[M])$-subalgebra of $F[M]$ generated by $F[M]^*$ is $F[M]$. So, we conclude that either M is abelian or $F[M]$ is an ore domain.

By a simple calculation we may obtain that $SL_n(D) \not\subset M$. If M is reducible, then $n > 1$, M contains an isomorphic copy of D^*. Thus, D' is torsion, and we obtain D = F and F is locally finite. If n > 2, then M contains an isomorphic copy of $GL_2(F)$. Therefore, M' contains an isomorphic copy of $SL_2(F)$. But this is a contradiction unless F is locally finite field. *So* we may assume n = 2, by a simple calculation, we have M' is an abelian group .

Now assume that M is irreducible, and so it is completely reducible. we conclude that M' is also completely reducible. Since M' is locally finite, F[M'] is a semisimple ring. As $M' \lhd M$, we conclude that

$$M \subseteq N_N(F[M']).$$

By maximality of M, two cases may occur, either $\mathrm{M} = \mathrm{N_N}(\mathrm{F}[\mathrm{M}'])$ or $\mathrm{N} = \mathrm{N_N}(\mathrm{F}[\mathrm{M}'])$. If $\mathrm{M} = \mathrm{N_N}(\mathrm{F}[\mathrm{M}'])$, then $\mathrm{F}[\mathrm{M}']^* \cap \mathrm{N} \subseteq \mathrm{M}$. Since M′ is locally finite, we obtain a contradiction. Consider $\mathrm{N} = \mathrm{N_N}(\mathrm{F}[\mathrm{M}'])$. We have either $\mathrm{M}' \subseteq \mathrm{F}$ or $\mathrm{F}[\mathrm{M}'] = \mathrm{M_n}(\mathrm{D})$. By our assumption the first case cannot happen. Now, assume that $\mathrm{F}[\mathrm{M}'] = \mathrm{M_n}(\mathrm{D})$.

Since M′ is a locally finite normal subgroup ofM, by Corollary 5.4.6 of [7], $\frac{\mathrm{M}}{\mathrm{C_M}(\mathrm{M}')}$ is locally finite and it has a metabelian normal subgroup of finite index. Since $\mathrm{F}[\mathrm{M}'] = \mathrm{M_n}(\mathrm{D})$., we have $\mathrm{C_M}(\mathrm{M}') = \mathrm{F}^*$. Therefore, $\frac{\mathrm{M}}{\mathrm{F}^*}$ has a metabelian normal subgroup of finite index. Suppose G is a normal subgroup of M such that $\frac{\mathrm{G}}{\mathrm{F}^*}$ is a metabelian normal subgroup of $\frac{\mathrm{M}}{\mathrm{F}^*}$ and $[\frac{\mathrm{M}}{\mathrm{F}^*}:\frac{\mathrm{G}}{\mathrm{F}^*}] < \infty$. Hence, we obtain that $[\mathrm{M}:\mathrm{G}] < \infty$ and $\mathrm{G}' \subseteq \mathrm{F}$.

G′ is abelian-by-finite. Thus, the group ring $F\mathrm{G}'$satisfies a polynomial identity. We conclude that F[G′] satisfies a polynomial identity, and hence D satisfies a polynomial identity. We have $[D:F] < \infty$. Since M is an absolutely irreducible skew linear group, we conclude that M is an irreducible linear group. Therefore, by Theorem 6 of [7, p. 135], M contains an abelian normal subgroup H, say, of finite index. If $H \subseteq \mathrm{F}^*$, then $\frac{\mathrm{M}}{\mathrm{F}^*}$ is finite. We arrive at a contradiction. So, H is non-central, which reduces to the previous case.

Now, consider that $\mathrm{G}' \subseteq \mathrm{F}$. Thus, G is nilpotent. With a similar argument as above, we obtain that either $\mathrm{M}' \cap G \subseteq \mathrm{F}^*$ or $F[\mathrm{M}' \cap G] = \mathrm{M_n}(\mathrm{D})$. $\mathrm{M}' \cap G$ is a locally finite nilpotent group. As above, the second case cannot happen. So, assume $\mathrm{M}' \cap G \subseteq \mathrm{F}^*$. Since $\mathrm{F}[\mathrm{M}'] = \mathrm{M_n}(\mathrm{D})$ and $[\mathrm{M}':\mathrm{M}' \cap G] < \infty$, we conclude that $[D:F] < \infty$. A similar argument as above holds.

3 ACKNOWLEDGEMENTS

The author thanks the Research Council of the University of Garmsar for support.

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