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A subnormally condition on certain number of elements of a group

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Abstract

Let $n > 0$ be an integer number. We say that a group G satisfies the condition $(\mathcal{S}, n)$ whenever in every subset X of G containing $n+1$ elements, there exist distinct elements x and y in X such that $\langle x\rangle$ and $\langle y\rangle$ are subnormal in $\langle x, y\rangle$. In this talk we study finite groups G in $(\mathcal{S}, n)$ and find a bound (depending only on n) for the size of every semisimple finite group satisfying the condition $(\mathcal{S}, n)$.

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1 Introduction

Let G be a group and let H be a subgroup of G. Then H is said to be subnormal in G, or simply H sn G, if there exists a finite series

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G.$$

The length of the shortest such series is called the defect of H in G. A group is called a Baer group if every cyclic subgroup is subnormal. If every cyclic subgroup of G is subnormal of defect at most n then we say that G is an n-Baer group or a B_n-group. Under a stronger hypothesis that every subgroup of G is subnormal of defect at most n, we say that G is a U_n-group. By a theorem of Roseblade [4], every U_n-group is nilpotent and the nilpotency class is bounded by a function only depended on n. this function is however still not well understood. Now for any two elements a and b of G, we define inductively $[a,_n b]$ the n-Engel commutator of the pair (a, b), as follows:

$$[a,_0 b] := a, \ [a, b] = [a,_1 b] := a^{-1}b^{-1}ab \text{ and } [a,_n b] = [[a,_{n-1} b], b] \text{ for all } n > 0.$$

An element x of G is called right (or left) Engel, provided that $[x,_n g] = 1$ (or $[g,_n x] = 1$) for all $g \in G$ and positive integer number $n = n(g)$. If every $x \in G$ is left Engel element, we say G is an Engel group. Now let $\langle x\rangle$ be subnormal in $\langle x, y\rangle$ of defect n. Then we have a series

$$\langle x\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = \langle x, y\rangle.$$

Therefore

$$[y, x] \in H_{n-1}, [y, x, x] \in H_{n-2}, \ldots, [y,_n x] \in H_0 = \langle x\rangle.$$

Thus $[y,_{k+1} x] = 1$. It is a well–known fact that for a finite group G, the following are equivalent:

1. G is nilpotent.

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2. Every cyclic subgroup of G is subnormal.

3. $\langle x\rangle$ sn $\langle x, y\rangle$ for all $x, y \in G$.

4. G is an Engel group.

The only difficult part, which is proved originally by M. Zorn [6], is the implication (4) $\Longrightarrow$ (1). Therefore every Baer group is Engel and if G is finite then G is a Baer group. Paul Erdos posed the following question [2]: Let G be an infinite group. If there is no infinite subset of G whose elements do not mutually commute, which called non–commuting subset of G, is there then a finite bound on the cardinality of each such set of elements? The affirmative answer to this question was obtained by B. H. Neumann who proved in [2] that a group is centre-by-finite if and only if every infinite subset of the group contains two different commuting elements. Payber [3] has shown that if the group G contains at most n pairwise non–commuting elements, then $|\frac{G}{Z(G)}| < c^n$ for some constant c. That means if every subset X of G containing $n+1$ elements contains tow distinct commuting element, then $|\frac{G}{Z(G)}| < c^n$ for some constant c. Now let $n > 0$ be an integer and $\mathcal{X}$ be a class of groups. We say that a group G satisfies the condition $(\mathcal{X}, n)$ whenever in every subset with $n+1$ elements of G there exist distinct elements x, y such that $\langle x, y\rangle$ is in $\mathcal{X}$. So if $\mathcal{A}$ is the class of abelian groups and $G \in (\mathcal{A}, n)$ then by [3] $|\frac{G}{Z(G)}| < c^n$ for some constant c.

Tomkinson in [5] proved that if G is a finitely generated soluble group satisfying the condition $(\mathcal{N}, n)$, whenever $\mathcal{N}$ is the class of nilpotent groups, then $|\frac{G}{Z^*(G)}| < n^{n^4}$, where $Z^*(G)$ is the hypercentre of G . This result gives a bound for the size of every finite soluble centerless group satisfying the condition $(\mathcal{N}, n)$. Abdollahi and Mohammadi Hassnabadi [1] study the class $(\mathcal{N}, n)$ and they gave a bound only depending on n for the size of finite semisimple groups in $(\mathcal{N}, n)$.

Now let G be a group and let $n > 0$ be an integer. We say that a group G is in the class $(\mathcal{S}, n)$ if every subset of G containing $n+1$ elements contains distinct elements x and y such that $\langle x\rangle$ and $\langle y\rangle$ are subnormal in $\langle x, y\rangle$. In this talk we study finite groups in the class $(\mathcal{S}, n)$ and we show that the size of finite semisimple group G in $(\mathcal{S}, n)$ is bounded by a function only depending on n. In fact we will see that $|G| < c^{2n^2[log_{60}n]}[log_{60}n]!$ when $G \in (\mathcal{S}, n)$ is a finite semisimple group.

2 Main results

By definition it is clear that $(\mathcal{N}, n) \subset (\mathcal{S}, n)$. As we mentioned above Abdollahi and Mohammadi Hassnabadi [1] have shown that $|G| < c^{2n^2[log_{21}n]}[log_{21}n]!$ whenever $G \in (\mathcal{N}, n)$ is a finite semisimple group. In this section we improve this bound for finite semisimple groups in $(\mathcal{S}, n)$. Before giving this bound we need some lemmas.

Lemma 2.1. *Let $G_1, \ldots, G_t$ are finite groups that $G_i \notin (\mathcal{S}, n_i)$, for some $1 \le i \le t$, and for $j \ne i$ $0 < n_j \le |G_j|$. Then $G_1 \times \cdots \times G_t$ is not in $(\mathcal{S}, n)$ where $n = n_1 \cdots n_t$.*

Remark 2.2. It is not difficult to see that if $G \in (\mathcal{S}, 1)$ then G is an Engel group and therefore G is nilpotent by a result of Zorn [6] if G is finite. Thus we have,

Corollary 2.3. *If $M_1, \ldots, M_t$ are finite simple groups, then $M_1 \times \cdots \times M_t$ is not in $(\mathcal{S}, 60^{t-1})$.*

Lemma 2.4. *Let G be a finite group in $(\mathcal{S}, n)$ and let p be a prime dividing $|G|$. Then if $p \ge n$, then $Z^*(G) \ne 1$.*

Corollary 2.5. *If $G \in (\mathcal{S}, n)$ is a finite simple group, then $|G| < c^{n^2}$ where $c < n$ is a constant.*

Theorem 2.6. *Let $G \in (\mathcal{S}, n)$ be a finite semisimple group. Then $|G| < c^{2n^2[log_{60}n]}[log_{60}n]!$.*

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