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On Schur's Theorem for Leibniz algebras

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Abstract

The concept of Some properties of the second homology and cover of Leibniz algebras are established. By constructing a stem cover, the second Leibniz homology and cover of abelian, Heisenberg Lie algebras and cyclic Leibniz algebras are described. Also, for the dimension of a non-cyclic nilpotent Leibniz algebra L, we obtain $dim(HL_2(L)) \geq 2$.

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1 Introduction

All algebras considered in this paper are finite dimensional over a field of characteristic different from 2. The terminology and notations employed agree with the standard usage as in [3]. Leibniz algebras are non-antisymmetric generalizations of Lie algebras. Loday (see [5, 6]) propounded a new type of algebras satisfying only Leibniz relations when he tried to formulate the non-commutative homology of a Lie algebra which is defined by replacing $\otimes$ by $\wedge$ in the Chevalley-Eilenberg complex of a Lie algebra. Recently, the theory of Leibniz algebras has been studied in some articles and several results of Lie algebras have been developed to Leibniz algebras. An algebra L over a field K is called a (left) Leibniz algebra if for any $a \in L$ the left multiplication map $l_a : L \to L$ given by $l_a(x) = [a, x]$ is a derivation, i.e. for all $x, y, z \in L$

$$[x, [y, z]] = [[x, y]z] + [y, [x, z]].$$

Obviously, if $[x, x] = 0$ for all $x \in L$, then a Leibniz algebra is a Lie algebra and Leibniz identity becomes the classical Jacobi identity.
It is well known that for a Leibniz algebra L, the space spanned by squares of elements, $Leib(L) = span\{[x, x]; x \in L\}$, is an ideal of L contained in the left center of L. Moreover, $Leib(L)$ is the minimal ideal of L with respect to the property that the quotient algebra $L/Leib(L)$ is a Lie algebra.
For any Leibniz algebra L, there is a tensor complex associated to L:

$$CL_*(L) : \cdots \to L^{\otimes n} \xrightarrow{d} L^{\otimes(n-1)} \xrightarrow{d} \cdots \xrightarrow{d} L \xrightarrow{0} K$$
$$d(x_1 \otimes \cdots \otimes x_n) := \sum_{1 \leq i \leq j \leq n} (-1)^i (x_1 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes [x_i, x_j] \otimes x_{j+1} \otimes \cdots \otimes x_n)$$

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The Leibniz homology (with trivial coeffcients) of L is defined as

$$HL_*(L) := H_*(CL_*(L), d).$$

The Leibniz homology of L can be interpreted as

$$HL_*(L) = Tor^*_{UL}(U(L/Leib(L)), K),$$

where $U(L/Leib(L))$ is the universal enveloping algebra of the quotient Lie algebra $L/Leib(L)$ and UL is the universal enveloping of the Leibniz algebra L. See [7] for more details. If L is a Leibniz algebra of dimension n, then the maximal possible dimension for $HL_i(L)$ is n^i which is met if and only if L is abelian. In the following proposition, we refine this inequality in the second step.

Proposition 1.1. *Let L be a n-dimensional Leibniz algebra. Then* $\dim(L^2) + \dim(HL_2(L)) \leq n^2$.

2 Stem cover of Leibniz algebras

Wiegold (1965) obtained an estimate for the order of commutator subgroup of a p-group G in terms of the order of $G/Z(G)$. Later, Batten (1993) in her dissertation obtained a similar result for Lie algebras. We start by establishing a parallel result for Leibniz algebra

Lemma 2.1. *Let L be a Leibniz algebra such that* $\dim(L/Z(L)) = n$ *then* $\dim(L^2) \leq n^2$.

Now, we go on to show that when equality holds in Lemma 2.1. We use the following notations through rest of the paper

$$Z^l = \{x \in L : [x, L] = 0\},$$
$$Z_2(L) = \{x \in L : [x, L], [L, x] \subseteq Z(L)\}.$$

Proposition 2.2. *Let L be a non-abelian nilpotent Leibniz algebra such that* $\dim(L/Z(L)) = n$ *and* $\dim(L^2) = n^2$ *then $L/Z(L)$ is a Lie algebra.*

Definition 2.3. For any integer n, let $L_n = span\{x_1, \cdots, x_n, x_{ij} : 1 \leq i, j \leq n\}$ be the (n^2+n)-dimensional Leibniz algebra with $[x_i, x_j] = x_{ij}$ for all $1 \leq i, j \leq n$ and all other products of basis elements being zero.

Proposition 2.4. *Let L be a non-abelian nilpotent Leibniz algebra such that* $\dim(L/Z(L)) = n$ *and* $\dim(L^2) = n^2$. *Then there exists an integers n such that $L \cong L_n \oplus A$, where A is a finite-dimensional abelian Lie algebra.*

Definition 2.5. Let $(e) : 0 \to N \to K \xrightarrow{\pi} L \to 0$ be a central extension of Leibniz algebras, then (e) (or π according to the notations of category theory) is called a stem extension of L if the induced morphism $HL_1(\pi) : HL_1(K) \to HL_1(L)$ is an isomorphism. Furthermore, (e) is called a stem cover if $HL_2(\pi)$ is zero.

Remark 2.6. If $(e) : 0 \to N \to K \xrightarrow{\pi} L \to 0$ is a stem extension of a finite-dimensional Leibniz algebra L then by Lemma 2.1, N and consequently K are also of finite dimensions. Similar to contexts of Lie algebras, (e) is called a maximal stem extension of L if $\dim(K)$ is maximal among all stem extensions of L.

Proposition 2.7. *Let $(e) : 0 \to N \to K \to L \to 0$ be a central extension of Leibniz algebras, then*

(i) (e) is a stem extension if and only if $N \subseteq L^2$.

(ii) If (e) is a stem cover, then (e) is isomorphic to (the unique class of) stem extension $0 \to HL_2(L) \to L^\circ \to L \to 0$.

(iii) Every stem extension of L is an epimorphic image of some stem cover.

Corollary 2.8. . *Let L be a finite-dimensional Leibniz algebra, then* $(e) : 0 \to N \to L^\circ \to L \to 0$ *is a stem cover of L if and only if L° has the maximal dimension among all stem extensions of L.*

Remark 2.9. Suppose L is a Lie algebra. The Lie algebra L^* is called a Lie cover of L if there exists an ideal $A \subseteq (L^*)^2 \cap Z(L^*)$ such that $A \cong H_2(L)$ and $L^*/A \cong L$, where $H_2(L)$ is the second Chevalley-Eilenberg homology of L. It is well known that L^* has maximal dimension among all stem extensions of L in the category of Lie algebras. Hence, besides Leibniz covers, we can think about Lie covers for a Lie algebra.

3 The second homology of nilpotent Leibniz algebras

Let $L = \langle a \rangle$ be a cyclic Leibniz algebra of dimension n and suppose $\{a, a^2 = [a, a], \cdots, a^i = [a, a^{i-1}], \cdots, a^n\}$ is a basis for L. It can be easily checked that $[a, a^n] = \alpha_2 a^2 + \cdots + \alpha_n a^n$ for some $\alpha_2, \cdots, \alpha_n \in K$. Note that if L is a nilpotent Leibniz algebra, then we should have $[a, a^n] = 0$. In the following proposition, we compute the second homology of a cyclic Leibniz algebra.

Proposition 3.1. *Let L be a cyclic Leibniz algebra of dimension n. Then* $\dim(HL_2(L)) = 1$.

Theorem 3.2. *Let L be a nilpotent Leibniz algebra then $HL_2(L)$ is nontrivial. In particular, if L is a nilpotent non-cyclic Leibniz algebra then* $\dim(HL_2(L)) \geq 2$.

Corollary 3.3. . *Let L be a two-step nilpotent Lie algebra. Then*

$$\dim(L/Z(L)) \leq \dim(HL_2(L)).$$

Now, we present the following general result to compare the Lie cover and Leibniz cover of a Lie algebra

Theorem 3.4. *Let L be a finite-dimensional Lie algebra and L^*, L° be the Lie cover and Leibniz cover of L, respectively. Then* $L^* \cong L^\circ/Leib(L^\circ)$.

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