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Irreducible relative central extension of leibniz algebras

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Abstract

In 1978, Loday introduced the concept of relative central extension of a pair of group (G, N). In this paper, we introduce the notion of irreducible relative central extension and it is applied in defining the primitive relative central extension and relative representation for a pair of leibniz algebras (L, N). We show that a pair of leibniz algebras (L, N) has at least one relative representation leibniz algebra. Finally, it is proved that every primitive relative central extension of a pair o leibniz algebras (L, N), in which $N = [N, L]$, is uniquely determined.

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1 Introduction

Let N be a ideal of a leibniz algebra L, then (L, N) is called a *pair* of leibniz algebras.

Definition 1.1. A relative central extension of a pair of leibniz algebras (L, N) is a homomorphism $\sigma : M \longrightarrow L$ accompanied with an action of L on the Leibniz algebra M, which is denoted by $[m, l] + [l, m]$ for all $m \in M$ and $l \in L$, and satisfies the following conditions:

(1) $\sigma(M) = N$;

(2) $\sigma[m, l] = [\sigma(m), l]$, for all $l \in L$ and $m \in M$;

(3) $\sigma[m, m'] = [\sigma(m), m'] + [m, \sigma(m')]$, for all $m, m' \in M$;

It is evident that the adjoint drivation $ad : N \longrightarrow L$ is defined by $ad(n) = ad_n(l) = [n, l]$ is a simple example of a relative central extension.

For each relative central extension $\sigma : M \longrightarrow L$, We use the following notations through rest of the paper

$$Z^l(L) = \{x \in L : [x, l] = 0, \forall l \in L\}$$
$$Z(M, L) = \{m \in M : [m, l] + [l, m] = 0, \forall l \in L\}.$$

Clearly, $\ker \sigma \subseteq Z(M, L)$. It is known that if $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ is a free presentation for a Leibniz algebra L, then $\mathcal{M}(L)$, the Schur multiplier of L, is isomorphic with $F^2 \cap R/[F, R]$. to any pair of leibniz algebras (L, N), when N has a complement in L as follows:

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Let $\mu : \mathcal{M}(L) \longrightarrow \mathcal{M}(\frac{L}{N})$ be a nature epimorphism then

$$\mathcal{M}(L, N) = \ker\Big(\mu : \mathcal{M}(L) \longrightarrow \mathcal{M}(\frac{L}{N})\Big),$$

is defined to be the Schur multiplier of a pair of leibniz algebras (L, N).

Now, if P is a ideal of F such that $N \cong P/R$, then one can see that

$$\mathcal{M}(L, N) \cong \frac{[P, F] \cap R}{[F, R]}.$$

if (L, N) is a pair of finite leibniz algebras then $\mathcal{M}(L, N)$ is a finite leibniz algebra. It is also independent of the choice of the presentation for L.

We always assume that the ideal N has a complement in L, and using the above notation all through the rest of the paper.

A relative central extension $\sigma : M \longrightarrow L$ is said to be the *epimorphic* (*isomorphic*) image of a relative central extension $\sigma_1 : M' \longrightarrow L$, whenever there exists an epimorphism (isomorphism) $\varphi : M \longrightarrow M'$. A relative central extension $\sigma : M \longrightarrow L$ is said to be *finite*, if L is a finite leibniz algebra.

Lemma 1.2. *Let (L, N) be a pair of leibniz algebras, $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ be a free presentation of L and P a ideal of F such that $N = P/R$. Then the map $\delta : P/[F, R] \longrightarrow F/R$ given by $\delta(x + [F, R]) = x + R$ is a relative central extension for the pair of leibniz algebras (L, N).*

Definition 1.3. Let $\sigma : M \longrightarrow L$ be a relative central extension of a pair of leibniz algebras (L, N) then

(i) σ is said to be *irreducible*, if there is no subalgebra L of M such that $H \cap \ker\sigma = L$.

(ii) The irreducible relative central extension σ of a pair of finite leibniz algebras (L, N) is called *primitive*, when

$$dim([M, L] \cap \ker\sigma) = dim\mathcal{M}(L, N).$$

(iii) A primitive relative central extension σ is said to be *relative representation leibniz algebra*, if $dimM = dimNdim\mathcal{M}(L, N)$ or equivalently $dim\ker\sigma = dim\mathcal{M}(L, N)$.

2 Main Results

Theorem 2.1. *Let $\sigma : M \longrightarrow L$ be an irreducible relative central extension and $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ be a free presentation of theleibniz algebra L. Then there exists a subalgebra $\bar{J} = J/[F, R]$ of $\bar{P} = P/[F, R]$ such that*

$$M \cong \frac{\bar{P}}{\bar{J}} \quad and \quad \ker\sigma \cong \frac{\bar{R}}{\bar{J}}.$$

Theorem 2.2. *The relative central extension $\sigma : M \longrightarrow L$ is irreducible if and only if the maximal subalgebras of M contain $\ker\sigma$, whenever they contain $[M, L]$.*

Theorem 2.3. *If $\sigma : M \longrightarrow L$ is an irreducible relative central extension of the pair of leibniz algebras (L, N), then*

$$dim[M, L] \cap \ker\sigma \leq dim\mathcal{M}(L, N).$$

In particular, σ is primitive if and only if $J \cap [F, P] = [F, R]$.

In the following useful result we show that every irreducible relative central extension of a pair of finite leibniz algebras (L, N), is the epimorphic image of a primitive relative central extension.

Theorem 2.4. *(a) If $\sigma : M \longrightarrow L$ is an irreducible relative central extension. Then, for some leibniz algebra M_0, there exists a primitive relative central extension $\sigma_0 : M_0 \longrightarrow L$ and a ideal $N_0 \subseteq \ker\sigma_0$ with*

$$M \cong \frac{M_0}{N_0} \quad and \quad dim\frac{[M, L]}{M} = dim\frac{[M_0, L]}{M_0}.$$

(b) *If $\sigma : M \longrightarrow L$ is an irreducible relative central extension, Then $\sigma_1 : M/K_1 \longrightarrow L$ is an irreducible relative central extension of the pair of leibniz algebras (L, N), for each subalgebra K_1 of $\ker\sigma$.*

Theorem 2.5. (a) *If $\sigma : M \longrightarrow L$ is a primitive relative central extension of the pair of finite leibniz algebras (L, N). Then*

$$[M, L] \cap \ker\sigma \cong \mathcal{M}(L, N).$$

In particular,

$$dimM \geq dimN dim\mathcal{M}(L, N).$$

(b) *The following conditions are equivalent:*

(i) *The extension $\sigma : M \longrightarrow L$ is a relative representation leibniz algebra.*

(ii) $\ker\sigma \subseteq [M, L]$.

(iii) $dim\frac{[M,L]}{M} = dim\frac{[N,L]}{N}$.

Corollary 2.6. *Every pair of finite leibniz algebras (L, N) has a relative representation leibniz algebra.*

Corollary 2.7. *If $\sigma_i : M_i \longrightarrow L$ $(i = 1, 2)$ is a primitive relative central extension of the pair of finite leibniz algebras (L, N), then the following conditions hold:*

(i) $[M_1, L] \cap \ker\sigma_1 \cong [M_2, L] \cap \ker\sigma_2 \cong M(L, N)$,

(ii) $[M_1, L] \cong [M_2, L]$.

Theorem 2.8. *Let (L, N) be a pair of finite leibniz algebras, in which $N = [N, L]$ and $\sigma : M \longrightarrow L$ be a relative central extension of L. Then the following statements hold:*

(a) *$\sigma_1 : [M, L] \longrightarrow L$ is an irreducible relative central extension of (L, N), where $\sigma_1 = \sigma\big|_{[M,L]}$.*

(b) *M is the direct sum of $[M, L]$ by $\ker\sigma$.*

(c) *The relative central extension $\sigma : M \longrightarrow L$ is irreducible if and only if $M = [M, L]$.*

(d) *The relative central extension $\sigma : M \longrightarrow L$ is primitive if and only if σ is a relative representation leibniz algebra of (L, N).*

(e) *Every primitive relative central extension $\sigma : M \longrightarrow L$ is uniquely determined. Also, every irreducible relative central extension of L is the epimorphic image of a relative representation leibniz algebra.*

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