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On the group-coset graphs

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Abstract

In this note, we study the group-coset graphs and review their properties. Among other results, we prove that every vertex-transitive graph is a group-coset graph and also we construct the Petersen graph as the group-coset graph with GAP [4].

Keywords: Group-coset graph, Cayley graph, Vertex-transitive graph, Symmetric graph
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1 Introduction

The family of Cayley graphs is of great importance in algebraic graph theory. Some families of graphs can be constructed as Cayley graphs but these are rare (see for example [2]). Cayley graphs are vertex-transitive but there are a large number of vertex-transitive graphs which are not Cayley. On the other hand, the Sabidussi's representation theorem states that a graph is vertex-transitive if and only if it is isomorphic to a group-coset graph (cf. [3, Theorem 4.3]). Therefore it is necessary to study these graphs because we can construct a vertex-transitive non-Cayley graph with a group-coset graph. The first example among vertex-transitive non-Cayley graphs is the Petersen graph which is a group-coset graph. As far as we know, the group-coset graph is a forgotten concept in the recent works in literature and therefore in this note we study these graphs and review their properties. For example, we construct the Petersen graph as the group-coset graph with GAP [4].

2 Preliminaries

In this note, all graphs are undirected and simple, i.e., there are no loops or multiple edges. Let G be a group, H a proper subgroup of this group, and S an inverse-closed subset of the group G such that $S \cap H = \varnothing$. Then the group-coset graph $\mathrm{GC(G, H, S)}$ is a graph whose vertex set is the set of all the right cosets of H in G and two vertices Hx and Hy are adjacent (denoted by $Hx \sim Hy$) whenever $xy^{-1} \in HSH$. It is trivial to see that if $H = \{e\}$, where e denotes the identity element of the group G, then the group-coset graph $\mathrm{GC(G, H, S)}$ is indeed the Cayley graph $\mathrm{Cay(G, S)}$.

Lemma 2.1. *The group-coset graph* $\mathrm{GC(G, H, S)}$ *is vertex-transitive.*

Proof. Similar as Cayley graphs, the collection of the automorphisms of this graph with the right multiplication is a subgroup of its automorphism group which acts transitively on the vertex set and this completes the proof. □

[1] speaker

Lemma 2.2. *Let Γ be a vertex-transitive graph. Then it is isomorphic to a group-coset graph.*

Proof. Let Γ be a vertex-transitive graph with vertex set $V(\Gamma)$ and G its automorphism group. Let x be an arbitrary fixed vertex of this graph, $H = G_x = \{\sigma \in G \mid \sigma(x) = x\}$, and $S = \{\sigma \in G \mid x \sim x\sigma\}$. Then the graph Γ is isomorphic to the group-coset graph $\mathrm{GC(G, H, S)}$. To see this, define

$$\varphi : V(\mathrm{GC(G, H, S)}) \to V(\Gamma)$$

$$\varphi(H\sigma) = x\sigma$$

It is trivial to see that φ is a well-defined bijective map. Moreover, if $H\sigma_1 \sim H\sigma_2$, then $\sigma_1\sigma_2^{-1} = h_1\sigma h_2$, where $h_1, h_2 \in H$ and $\sigma \in S$. This implies that $h_1^{-1}\sigma_1\sigma_2^{-1}h_2^{-1} = \sigma$. It follows that $x \sim xh_1^{-1}\sigma_1\sigma_2^{-1}h_2^{-1}$ and therefore $x \sim x\sigma_2\sigma_1^{-1}$. Hence $x\sigma_1 \sim x\sigma_2$ and this completes the proof. □

Example 2.3. The Petersen graph with vertex set $\{1, 2, \dots, 10\}$ is vertex-transitive and therefore it is a group-coset graph. The automorphism group of this graph is isomorphic to the symmetric group $G = S_5$. By using GAP [4], $G = \langle(2, 5)(3, 6)(4, 7), (2, 3)(5, 6)(9, 10), (3, 4)(6, 7)(8, 9), (1, 2)(6, 8)(7, 9)\rangle$, the stabilizer G_1 is $H = \langle(2, 5)(3, 6)(4, 7), (2, 3)(5, 6)(9, 10), (3, 4)(6, 7)(8, 9)\rangle$ which is isomorphic to the dihedral group of order 12. It turns out that if $S = \{(1, 8)(3, 5)(4, 9)(7, 10)\}$, then the Petersen graph is isomorphic to the group-coset graph $\mathrm{GC(G, H, S)}$.

3 Some properties of a group-coset graph

Suppose that G is a group and H its proper subgroup. Suppose further that $S = \{u, u^{-1}\}$, where $u \in G \setminus H$. Then the group-coset graph $\mathrm{GC(G, H, S)}$ is a symmetric graph. Recall that a symmetric graph is a vertex-transitive and edge-transitive graph. A group-coset graph is vertex-transitive and therefore to conclude this result, it is sufficient that we prove the group-coset graph $\mathrm{GC(G, H, S)}$ is edge-transitive. Let $\{H, Hx\}$ and $\{H, Hy\}$ are two edges. Then $x, y \in HSH$ and therefore we have to consider only two cases.
Case 1. Let $x = h_1uh_2$ and $y = h_3uh_4$, where $h_i \in H$ for $i = 1, 2, 3, 4$. Then $y = h_3h_1^{-1}xh_2^{-1}h_4$ and therefore $Hy = Hxh_2^{-1}h_4$ and we can easily define an automorphism on the group-coset graph with the right multiplication which maps $\{H, Hx\}$ to $\{H, Hy\}$.
Case 2. Let $x = h_1uh_2$ and $y = h_3u^{-1}h_4$, where $h_i \in H$ for $i = 1, 2, 3, 4$. Then $Hxg = H$ and $Hg = Hy$, where $g = h_2^{-1}u^{-1}h_4$ as desired.

Theorem 3.1. *Let Γ be a symmetric graph. Then it is isomorphic to the group-coset graph* $\mathrm{GC(G, H, S)}$ *in which $S = \{\sigma, \sigma^{-1}\}$, where $\sigma \in G \setminus H$.*

Proof. By Lemma 2.2, it is sufficient to prove that $S = \{\sigma, \sigma^{-1}\}$, where $\sigma \in G \setminus H$. If $\alpha \in S$, then $\{x, x\alpha\}$ is an edge and therefore there exists $\varphi \in G$ such that $\{x, x\sigma\}\varphi = \{x, x\alpha\}$ since this graph is edge-transitive. If $x\varphi = x$, then $\varphi \in H$ and therefore $H\sigma H = H\alpha H$. On the other hand, if $x\sigma\varphi = x$ and $x\varphi = x\alpha$, then $\sigma\varphi \in H$ and $\varphi\alpha^{-1} \in H$. Therefore $H\sigma H = H\alpha^{-1}H$ and this completes the proof. □

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