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Some properties of the $k-$extension of $p-$Fibonacci sequence

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Abstract

In this paper, By considering the $k-$extension of $p-$Fibonacci sequence, we obtain combinatorial identities. Also, by using the Riordan method, we get two factorizations of the Pascal matrix involving $k-$extension of $p-$Fibonacci sequence.

Keywords: $k-$extension of $p-$Fibonacci sequence, Riordan arrays, Pascal matrix.
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1 Introduction

The Fibonacci sequence is defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}, n \geq 3$, with the initial values $f_1 = f_2 = 1$.The Fibonacci sequence and its generalization Fibonacci number sequence have been studied for example in [2, 4]. This sequence has been extended in many ways. One such extension that will be used in this paper are the $p-$Fibonacci sequence and the k-extension of the p-Fibonacci sequence (see [3]).

Definition 1.1. For $k, p \geq 1$, the k-extension of the p-Fibonacci sequence $\{f^p(k,n)\}_{-\infty}^{\infty}$ is given by the following recurrence relation:

$$f^p(k,n) = \begin{cases} 0, & n < 1, \\ 1, & n = 1, \\ kf^p(k,n-1) + f^p(k,n-p-1), & n > 1. \end{cases}$$

For example if $k = 1$ and $p = 2$, we have $f^2(1,n) = f^2(1,n-1) + f^2(1,n-3)$ and $\{f^2(1,n)\}_{-\infty}^{\infty} = \{\ldots, 0, 1, 1, 1, 2, 3, 4, 6, 9, \ldots\}$.
The $n \times n$ lower triangular Pascal matrix, denoted by $P_n = [p_{ij}]$, is defined as follows [1]:

$$p_{ij} = \begin{cases} \binom{i-1}{j-1}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

The Riordan group was introduced in [5] as follows.
Let $R = [r_{ij}]_{i,j\geq 0}$be an infinite matrix with complex entries. Let $C_i(t) = \sum_{n\geq 0}^{\infty} r_{n,i}t^n$ be the generating function of the ith coloum of R. We call R a Riordan matrix if $c_i(t) = g(t)[f(t)]^i$, where

$$g(t) = 1 + g_1t + g_2t^2 + g_3t^3 + \cdots, \qquad f(t) = t + f_2t^2 + f_3t^3 + \cdots.$$

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In this case we write $R = (g(t), f(t))$ and denote by R the set of Riordan matrices. Then the set R is a group under matrix multiplication $*$, with the following properties:
$(i)(g(t), f(t)) * (h(t), l(t)) = (g(t)h(f(t)), l(f(t)))$,
$(ii)I = (1, t)$ is the identity element,
(iii) the inverse of R is given by $R^{-1} = (\frac{1}{g(\bar{f}(t)}, \bar{f}(t))$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$, that is,$f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

2 Main results

In this section, first we obtain some results from the k-extension of the p-Fibonacci sequence. Then, we get two factorizations of the Pascal matrix involving $k-$extension of $p-$Fibonacci sequence.

Now, we define the $n \times n$ k-extension of the p-Fibonacci matrix ($p \geq 2$), denoted by $F^p_{(k,n)} = [f^p_{(k,ij)}]$, as follows:

$$f^p_{(k,ij)} = f^p(k, i - j + 1).$$

Theorem 2.1. *For the inverse of the k-extension of the p-Fibonacci matrix, denoted by* $(F^p_{(k,n)})^{-1} = [f'^p_{(k,ij)}]$, *we have*

$$f'^p_{(k,ij)} = \begin{cases} 1, & \text{if } i = j, \\ -k, & \text{if } j = i - 1, \\ -1, & \text{if } j = i - (p+1), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. To find the inverse of the k-extension of the p-Fibonacci matrix, we define the $n \times n$ matrix $U^p_{(k,n)} = [u^p_{k,ij}]$ as follows:

$$u^p_{k,ij} = \begin{bmatrix} f^p(k,1) & 0 & 0 & \dots & 0 \\ f^p(k,2) & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ f^p(k,n) & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Clearly, $U^p_{(k,n)}$ is invertible and

$$(U^p_{(k,n)})^{-1} = \begin{bmatrix} f^P(k,1) & 0 & 0 & \dots & 0 \\ -f^P(k,2) & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ -f^P(k,n) & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Hence,

$$F^p_{(k,n)} = U^p_{(k,n)} \times (I_1 \oplus U^p_{(k,n-1)}) \times (I_2 \oplus U^p_{(k,n-2)}) \times \cdots \times (I_{n-2} \oplus U^p_{(2,2)}),$$

where I_j is an identity matrix. Since $(I_t \oplus U^p_{(k,n-t)})^{-1} = I_t \oplus (U^p_{(k,n-t)})^{-1}$, we have

$$(F^p_{(k,n)})^{-1} = (I_{n-2} \oplus (U^p_{(k,2)})^{-1}) \times \cdots \times (I_1 \oplus (U^p_{(k,n-1)})^{-1}) \times (U^p_{(k,n)})^{-1}.$$

Therefore,

$$f'^p_{(k,ij)} = \begin{cases} 1, & \text{if } i = j, \\ -k, & \text{if } j = i - 1, \\ -1, & \text{if } j = i - (p+1), \\ 0, & \text{otherwise.} \end{cases}$$

□

Theorem 2.2. *Let $g_f(t)$ be the generating function of the $k-$extension of the p-Fibonacci sequence. Then,*

$$g_f(t) = \frac{t}{1-kt-t^{p+1}}.$$

Proof. Let $g_f(t)$ be the generating functions of the k-extension of the p-Fibonacci sequence $f^p(k,n)$, then

$$\begin{aligned} g_f(t) &= \sum_{n=0}^{\infty} f^p(k,n)t^n \\ &= t + kt^2 + \sum_{n=3}^{\infty} kf^p(k,n-1)t^{n-1} + \sum_{n=3}^{\infty} f^p(k,n-p-1)t^{n-p-1} \\ &= t + kt^2 + k + \sum_{n=2}^{\infty} kf^p(k,n)t^n + t^{p+1} \sum_{n=3-p+1}^{\infty} f^p(k,n)t^n \\ &= t + kt^2 + kt(g_f(t) - t) + t^{p+1}g_f(t). \end{aligned}$$

By taking $g_f(t)$ parenthesis we get

$$g_f(t) = \frac{t}{1-kt-t^{p+1}}.$$

□

Lemma 2.3. *For $k,\ p \geq 1$, the k-extension of the p-Fibonacci sequence have the following exponetial representating:*

$$g_f(t) = t\exp(\sum_{n=1}^{\infty} \frac{t^n}{n}(k+t^p)^n).$$

Proof. Since

$$lng_f(t) = lnt - ln(1-kt-t^{p+1}),$$

and we have

$$-ln(1-kt-t^{p+1}) = -(-t(k+t^p) - \frac{t^2}{2}(k+t^p)^2 - \frac{t^n}{n}(k+t^p)^n - \cdots).$$

Therefore,

$$ln\frac{g_f(t)}{t} = \exp(\sum_{n=1}^{\infty} \frac{t^n}{n}(k+t^p)^n).$$

□

Here, we define an infinite $k-$extension of the p-Fibonacci matrix.

Definition 2.4. For $k,\ p \geq 1$, the k-extension of the p-Fibonacci matrix, denoted by $F(x) = [F^p(k,n)]$, as follows:

$$F(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ k & 1 & 0 & 0 & 0 & \cdots \\ k^2 + f^p(k,3-p-1) & k & 1 & 0 & 0 & \cdots \\ k^3 + f^p(k,4-p-1) & k^2 + f^p(k,3-p-1) & k & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} = (g_{F(x)}(t), f_{F(x)}(t)).$$

The matrix $F(x)$ is an element of the set of Riordan matrices. Since the first column of $F(x)$ is

$$(1, k, k^2 + f^p(k, 3-p-1), k^3 + f^p(k, 4-p-1), k^2 + f^p(k, 3-p-1), \cdots)^T.$$

Then it is obvious that $g_{F(x)}(t) = \sum_{n=0}^{\infty} f^p(k, n-p-1)t^n = \dfrac{t}{1-kt-t^{p+1}}$. In the matrix $F(x)$ each entry has a rule the upper two rows, that is,

$$f^p(k,n) = \begin{cases} 0, & n < 1, \\ 1, & n = 1, \\ kf^p(k,n-1) + f^p(k,n-p-1), & n > 1. \end{cases}$$

Then $f_{F(x)}(t) = t$, that is

$$F(x) = (g_{F(x)}(t), f_{F(x)}(t)) = (\frac{1}{1-kt-t^{p+1}}, t),$$

hence $F(x)$ is R. For these factorization, we need to define matrix $M(x) = (m_{ij}(x))$, as follows:

$$m_{ij}(x) = \binom{i-1}{j-1} - k\binom{i-2}{j-1} - \binom{i-(p+2)}{j-1}, \tag{1}$$

we have the infinite matrix $M(x)$ as follows:

$$M(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1-k & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{2}$$

Theorem 2.5. *Let $M(x)$ be the infinite matrix as* (2.2) *and $F(x)$ be the infinite $k-$extension of the p-Fibonacci matrix. Then $P(x) = F(x) * M(x)$, where P is the Pascal matrix.*

Proof. From the definitions of the infinite Pascal matrix and the infinite $k-$extension of the p-Fibonacci matrix we have the following Riordan representing

$$P(x) = (\frac{1}{1-t}, \frac{t}{1-t}), \quad F(x) = (\frac{1}{1-kt-t^{p+1}}, t).$$

Now we can find the Riordan representation of infinite matrix

$$M(x) = (g_{M(x)(t)}, f_{M(x)}(t))$$

as follows:

$$M(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1-k & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

From the first colum of the matrix $M(x)$ we obtain $M(x) = F^{-1}(x) * P(x)$ and

$$F^{-1}(x) = (g_{F(x)}(t), f_{F(x)}(t))^{-1} = (1-kt-t^{p+1}, t),$$

we have

$$M(x) = (\frac{1-kt-t^{p+1}}{1-t}, \frac{t}{1-t}),$$

complete the proof. □

Now we define the matrix $B(x) = (b_{ij}(x))$ as follows

$$b_{ij}(x) = \binom{i-1}{j-1} - k\binom{i-1}{j} - \binom{i-1}{j+p},$$

we have the infinite matrix $B(x)$ as follows

$$B(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1-k & 1 & 0 & 0 & \cdots \\ 1-2k-\binom{2}{1+p} & 1-k & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{3}$$

Lemma 2.6. *Let $B(x)$ be the matrix in (2.3), we have $P = B(x) * F(x)$.*

Proof. The proof is similar to that of Theorem 2.6. □

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