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A new version of isoclinism

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Abstract

Let G be a group and $\mathrm{IA}(G)$ denote the group of all automorphisms of G which induces the identity mapping on the abelianization (i.e. G/G'). Let $\mathrm{IA}_z(G)$ be the subgroup of $\mathrm{IA}(G)$ such that fix the center element-wise. In this paper, we introduce the new concept of IA-isoclinism between two groups (which is a new version of isoclinism) and prove similar results to the case of isoclinism including an application towards g-autocommuting probability of finite groups.

Keywords: Automorphisms; Isoclinism; IA-isoclinism
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1 Introduction

Let α be an automorphism of a given group G, we call α an IA-*automorphism* if $x^{-1}\alpha(x) \in G'$, for each $x \in G$. This concept was introduced by Bachmuch [1], in 1965. The set of all IA-automorphisms that fix the centre element-wise forms a normal subgroup of $\mathrm{IA}(G)$ and is denoted by $\mathrm{IA}_z(G)$ (see [4, 12, 14] for more information). The concept of isoclinism between two groups G and H was introduced by Hall [5], which is an equivalence relation among all groups and it is weaker than isomorphism. In fact, two groups G and H are *isoclinic* if and only if there exist isomorphisms $\alpha : G/Z(G) \to H/Z(H)$ and $\beta : G' \to H'$ so that β is induced by α, which are compatible. Clearly, all abelian groups are isoclinic with the trivial group. Hekster in [7, 8] studied the concept of n-isoclinism and isologism which are generalizations of isoclinism between groups. An interesting application of n-isoclinism of finite groups can be found in [10]. More generalizations of isoclinism between groups along whit applications can be found in [2, 9, 11, 13].
For each element $g \in G$ and $\alpha \in Aut(G)$, $[g, \alpha] = g^{-1}g^{\alpha} = g^{-1}\alpha(g)$ is the *autocommutator* of g and α. In [6], Hegarty introduced the following subgroups,

$$K(G) = \langle [g, \alpha] \mid g \in G, \alpha \in Aut(G)\rangle,$$

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \forall \alpha \in Aut(G)\},$$

which are called *autocommutator subgroup* and *absolute centre* of G, respectively. Clearly, they are both characteristic subgroups in G so that $K(G)$ contains the derived subgroup, G', and $L(G)$ is contained in, $Z(G)$, the centre of G.
We know that, two groups G and H are *isoclinic* (denoted by $G \sim H$), if there exist isomorphisms $\alpha : G/Z(G) \to H/Z(H)$ and $\beta : G' \to H'$ such that the following diagram is commutative:

$$\begin{array}{ccc} \frac{G}{Z(G)} \times \frac{G}{Z(G)} & \xrightarrow{\alpha\times\alpha} & \frac{H}{Z(H)} \times \frac{H}{Z(H)} \\ f_1 \downarrow & & \downarrow f_2 \\ G' & \xrightarrow{\beta} & H' \end{array}$$

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where,
$f_1(g_1Z(G), g_2Z(G)) = [g_1, g_2]$ and $f_2(h_1Z(H), h_2Z(H)) = [h_1, h_2]$, for each $h_i \in \alpha(g_iZ(G))$, $i = 1, 2$, and β induced by α.

In the next section we introduce the concept of IA-*isoclinism* between two groups G and H, which is a new version of isoclinism. Finally, some examples will be presented to specify the relation among the concepts of isoclinism, isomorphism and IA-isoclinism including an application towards g-autocommuting probability of finite groups.

2 Main results

In this section, we introduce the concept of IA-homoclinism (IA-isoclinism) between two groups G and H, which is a new version of homoclinism (isoclinism). We have

$$\mathrm{IA}_z(G) = \{\alpha \in IA(G) \ : \ \alpha(z) = z, \ \forall z \in Z(G)\}.$$

Consider the following subgroup of the autocommutator subgroup $K(G)$,

$$K_z(G) = \langle [g, \alpha] \mid g \in G, \alpha \in \mathrm{IA}_z(G)\rangle = [G, \mathrm{IA}_z(G)],$$

which is a normal subgroup of G'. On the other hand $G' = [G, \mathrm{Inn}(G)] \leq [G, \mathrm{IA}_z(G)]$, and so $G' = [G, \mathrm{IA}_z(G)]$.

In the following, we consider this form of G'.

Definition 2.1. For arbitrary groups G and H, let $\gamma : \mathrm{IA}_z(G) \to \mathrm{IA}_z(H)$, $\beta : G' \to H'$ and $\alpha = \gamma|_{Inn(G)} : G/Z(G) \to H/Z(H)$ be homomorphisms in such a way that the following diagram is commutative:

$$\begin{array}{ccc} \frac{G}{Z(G)} \times \mathrm{IA}_z(G) & \stackrel{\alpha\times\gamma}{\longrightarrow} & \frac{H}{Z(H)} \times \mathrm{IA}_z(H) \\ l_1 \downarrow & & \downarrow l_2 \\ G' & \stackrel{\beta}{\longrightarrow} & H' \end{array}$$

such that,
$(\alpha \times \gamma)(gZ(G), \theta) = (\alpha(g)Z(H), \gamma(\theta)) = (hZ(H), \gamma(\theta))$, $l_1(gZ(G), \gamma_1) = [g, \gamma_1]$, $l_2(hZ(H), \gamma_2) = [h, \gamma_2]$ and $\beta([g, \theta]) = [h, \gamma(\theta)]$, where $h \in \alpha(g)Z(H)$. Then $(\alpha \times \gamma, \beta)$ is said to be an IA-*homoclinism* and G and H are called IA-*homoclinic.*

Clearly, the above notion is a generalized version of homoclinism, if one replaces $\mathrm{IA}_z(X)$ by $\mathrm{Inn}(X)$, for any group X (see Hall [5]).

The following proposition shows that how one can define the *kernel* and *image* of IA-homoclinism between two groups.

Proposition 2.2. *Let* $(\alpha \times \gamma, \beta)$ *be a pair of* IA-*homoclinism between the groups* G *and* H*. Then*
(i) if $\alpha \times \gamma$ *is surjective, then so is* β*;*
(ii) if β *and* γ *are injective, then so is* $\alpha \times \gamma$*.*

Let $(\alpha \times \gamma, \beta)$ be an IA-homoclinism between the groups G and H. Then $Ker(\alpha \times \gamma, \beta) = Ker\beta$ and $Im(\alpha \times \gamma, \beta) = (H_1/Z(H), A_1)$ are defined to be the *kernel* and *image* of $(\alpha \times \gamma, \beta)$, where H_1 and A_1 are subgroups of H and $\mathrm{IA}_z(H)$, respectively. Definition We also say that $(\alpha \times \gamma, \beta)$ is an IA-*isoclinism* between G and H if $\alpha \times \gamma$ is surjective and β and γ are injective. In this case, we say that G and H are IA-*isoclinic* and denoted by $G \sim_{ia} H$.

One can easily observe that any abelian group is IA-isoclinic with the trivial group, and similarly the Dihedral and Quaternion groups of order 2^n are IA-isoclinic.
Rai [12] proved that if two finite groups G and H are isoclinic, then $\mathrm{IA}_z(G) \cong \mathrm{IA}_z(H)$. By using this result the following corollaries are obvious.

Theorem 2.3. *For arbitrary groups G and H, $G \sim H$ if and only if $G \sim_{ia} H$.*

Corollary 2.4. *If G be a group and A is an arbitrary abelian group, then $G \sim_{ia} G \times A$.*

In the following, under some condition, we show that each group G is IA-isoclinic with its factor group.

Theorem 2.5. *For a characteristic subgroup N of a given group G, if $N \cap G' = \langle 1 \rangle$ then $\mathrm{IA}_z(G) \cong \mathrm{IA}_z(G/N)$.*

Now, using the above theorem we are able to deduce the following corollaries.

Corollary 2.6. *Let N be a characteristic subgroup of a given finite group G such that N has trivial intersection with the subgroup G'. Then $G \sim_{ia} G/N$.*

Corollary 2.7. *Let M and N be two characteristic subgroups of a given finite group G such that $N \cap G' = 1$, then $G \sim_{ia} \frac{G}{N \cap M}$.*

The following result gives a useful criterion for two groups to be IA-isoclinic.

Proposition 2.8. *Let $A \leq Z(G)$, $B \leq Z(H)$ and $\gamma : \mathrm{IA}_z(G) \to \mathrm{IA}_z(H)$, $\beta : G' \to H'$, $\alpha : G/A \to H/B$ are isomorphisms so that $\beta([g, \theta]) = [h, \sigma]$, for all $g \in G$, $\theta \in \mathrm{IA}_z(G)$ and some $\sigma \in \mathrm{IA}_z(H)$ such that $h \in \alpha(gA)$ and $\gamma(\theta) = \sigma$. Then $G \sim_{ia} H$.*

Remark 2.9. By direct calculation we obtain $\mathbb{Z}_3 \sim_{ia} \mathbb{Z}_6$, while $\mathbb{Z}_3 \times \mathbb{Z}_2 \not\sim_{ia} \mathbb{Z}_6 \times \mathbb{Z}_2$. Therefore the product of two IA-isoclinic groups is not IA-isoclinic, in general.

The following result provides a sufficient condition so that the product of two IA-isoclinic groups is IA-isoclinic.

Corollary 2.10. *Let G_1, G_2, H_1 and H_2 be non-abelian finite groups with $G_i \sim_{ia} H_i$ for $i = 1, 2$. If G_i's and H_i's have mutually co-prime orders, then $G_1 \times G_2 \sim_{ia} H_1 \times H_2$.*

Now we give an application of IA-isoclinism towards g-autocommuting probability of finite groups, which was introduced by Dutta and Nath [3] recently in 2018. Recall that g-autocommuting probability of a finite group G is given by the ratio

$$Pr_g(G, Aut(G)) = \frac{|\{(x, \alpha) \in G \times Aut(G) : \ [x, \alpha] = g\}|}{|G||Aut(G)|}, \qquad (\star)$$

where $g \in G$. We conclude the paper with the following result.

Proposition 2.11. *Let G and H be two finite groups and $(\alpha \times \gamma, \beta)$ an IA-isoclinism between them. If G and H are perfect groups then*

$$Pr_g(G, Aut(G)) = Pr_{\beta(g)}(H, Aut(H)).$$

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