14th Iranian International Group Theory Conference

On the p-solvability of a finite group and its co-degrees of irreducible characters

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Abstract

Let G be a finite group and p be a prime number. The co-degree of an irreducible character χ of G is defined as $\chi^c(1) = \frac{[G:\mathrm{Ker}\chi]}{\chi(1)}$. In this talk, we show that a p-solvable group as G, is solvable if for every non-principal irreducible character χ of G, $p \mid \chi^c(1)$. Also, we prove that the p-parts of co-degrees of non-principal irreducible characters of G are same if and only if G is an elementary abelian p-group. Next, we show that if G is a p-solvable group and e is a positive integer such that $p^{e+1} \nmid \chi^c(1)$, for every irreducible character χ of G, then the p-length of G is not greater than e. Finally, we consider the case when p^2 does not divide the co-degrees of irreducible characters of G.

Keywords: co-degree of a character, p-solvable group, p-length

Mathematics Subject Classification [2010]: 20C15, 20D10, 20D05

1 Introduction

Throughout this paper, G is a finite group, p is a prime number and $\mathrm{Irr}(G)$ shows the set of irreducible characters of G. For every character $\chi \in \mathrm{Irr}(G)$, the number $\chi^c(1) = \frac{[G:\mathrm{Ker}\chi]}{\chi(1)}$ is called the co-degree of χ (see [8]). Some properties of $\chi^c(1)$ have been studied in [1, 2, 3, 4, 8]. Set $\mathrm{Codeg}(G) = \{\chi^c(1) : \chi \in \mathrm{Irr}(G)\}$. If n is a positive integer, n_p denotes the p-part of n. Also, $\pi(G)$ shows the set of prime divisors of order of G. For a p-solvable group G, the p-length of G, denoted by $\ell_p(G)$, is the minimum possible number of factors that are p-groups in any normal series of G which every factor is either a p-group or a p'-group.

J.G. Thompson proved that if the degree of every nonlinear irreducible character of G is divisible by p, then G has a normal p-complement (see[10]). Qian et al. showed that for every $p \in \pi(G)$, there exists $\chi \in \mathrm{Irr}(G)$ such that $p \mid \chi^c(1)$ (see [8]). The authors in [5] proved that G is an elementary abelian p-group if and only if $\mathrm{Codeg}(G) = \{1, p\}$. Alizadeh et al. proved that G is an elementary abelian group if and only if the set of co-degrees of irreducible characters of G be a two-member set (see [2]).

According to the result obtained in [1], the p-length of a p-solvable group is not greater than the number of the distinct co-degrees of its irreducible characters which are divisible by p.

Ito-Michler theorem says that if $p \in \pi(G)$ and for every irreducible character χ of G, $\chi(1)_p < p$, then $[G : O_p(G)]_p < p$. Lewis et al. have generalized Ito-Michler theorem in [7] and showed that if $p \in \pi(G)$ is an odd prime and every irreducible character χ of G satises $\chi(1)_p \leqslant p$, then $[G : O_p(G)]_p \leqslant p^4$. Qian got the smaller bound p^3 for $[G : O_p(G)]_p$, when G has no irreducible character of degree divisible by p^2 (see [9]).

[1] speaker

2 Main results

In [3], we have shown that:

Theorem 2.1. *Let G be a p-solvable group.*

(i) *If $p \mid \chi^c(1)$ for every non-principal character $\chi \in \mathrm{Irr}(G)$, then G is solvable.*

(ii) *Suppose that p is neither 2 nor a Mersenne prime. Then $p \mid \chi^c(1)$ for every non-principal character $\chi \in \mathrm{Irr}(G)$ if and only if G is a p-group.*

Theorem 2.2. *For every non-principal irreducible character χ of G, $\chi^c(1)_p = p^e$ if and only if $e = 1$ and G is an elementary abelian p-group.*

Next theorems have been proven in [4]:

Theorem 2.3. *If G is a p-solvable group and $p^{e+1} \nmid \chi^c(1)$, for every $\chi \in \mathrm{Irr}(G)$, then $\ell_p(G) \leqslant e$.*

Theorem 2.4. *If $\chi^c(1)_p \leqslant p$, for every irreducible character χ of G, then either $|G|_p = p$ or G is a p-solvable group of p-length one.*

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