The 2-pell number sequence of Fibonacci group F(r,2)

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ABSTRACT

In this paper we consider the Fibonacci group F(r,2), in which r odd, defined by the presentation:

$$F(r,2)=\left\langle a_1.a_2 \middle| (a_1a_2)^{\frac{(r-1)}{2}} = a_2a_1^{-1}, (a_2a_1)^{\frac{(r-1)}{2}} = a_1a_2^{-1} \right\rangle$$

Then we obtain the period of the generalized order 2-pell sequence of this group.

KEYWORDS: period, generalized order 2-pell sequence, Fibonacci group F(r,2)

1 INTRODUCTION

The k-Pell number sequence $\{P_n^k\}_{n=0}^{\infty}$ is defined by:

$$P_n^k = 2P_{n-1}^k + P_{n-2}^k + \cdots + P_{n-k}^k \quad n \geq k; \tag{1}$$

with initial conditions $P_0^k = 0. P_1^k = 0. \dots . P_{k-2}^k = 0$ *and* $P_{k-1}^k = 1$. We use $hP_k(m)$ to denoted the minimal length of the period of the k-Pell number sequence $\{P_n^k\ (mod\ m)\}_{n=0}^{\infty}$. The generalized order k-Pell sequence and their properties have been studied by several authors, for example see [1, 2, 3, 4, 5]. For example: $P_n^2 = 2P_{n-1}^2 + P_{n-2}^2$. Then $\{P_n^2\}_{n=0}^{\infty}$ ={0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, 470832, 1136689, 2744210, 6625109, 15994428, …} and $\{\{P_n^2\}(mod\ 5)\} = \{0.1.2.0.2.4.0. \dots\}$.

We now discuss the period of the generalized order 2-pell sequence of the Fibonacci group $F(r,2)$.

Definition1.1. Let X=$\{a_0.a_1. \dots . a_{j-1}\}$ and G=$< X >$ be a finite group. A generalized order k-Pell sequence $\{x_n\}_{n=0}^{\infty}$ of G is defined as follows:

$$x_n = \begin{cases} a_n & n \leq j-1 \\ x_1x_2 \dots (x_{n-1})^2 & j-1 < n \leq k \\ x_{n-k} \dots (x_{n-1})^2 & n > k. \end{cases} \tag{2}$$

Here, we discuss the period of the generalized order 2-Pell sequence of the Fibonacci groups $F(r,2)$, in which r odd, defined by the representation

$$F(r,2)=\left\langle a_1.a_2 \middle| (a_1a_2)^{\frac{(r-1)}{2}} = a_2a_1^{-1}, (a_2a_1)^{\frac{(r-1)}{2}} = a_1a_2^{-1} \right\rangle \tag{3}$$

Lemma 1.1. The relations of $F(r,2)$ imply the relation $a_1^2 = a_2^2$.

Proof. See [6].

2 THE FIBONACCI LENGTH OF *F*(*R*,2)

Now we find a standard form for the 2-Pell sequence $x_0, x_1,\ldots$ of *F*(*r*,2).

Lemma 2.1. Let $x_0, x_1,\ldots$ be the 2-Pell sequence of *F*(*r*,2), with $x_0 = a_1, x_1 = a_2$. Then for every *w*≥ 2 , we have

$$x_w = \begin{cases} a_1^{P_{w+1}-P_w} & w \equiv 0\ (mod\ 2) \\ a_2^{P_w+P_{w-1}} & w \equiv 1\ (mod\ 2) \end{cases} \tag{4}$$

Proof. We use induction on w. We have $x_0 = a_1 = a_1^{P_1-P_0}. x_1 = a_2 = a_2^{P_1+P_0}$.

Now assume that the result holds for all integers less than *w*, we consider two cases as following:

1) Let $w \equiv 1\ (mod\ 2)$. Then we have $x_{w-2} = a_2^{P_{w-2}+P_{w-3}},\ x_{w-1} = a_1^{P_w-P_{w-1}}$.

Using the fact that the relation $a_1^2 = a_2^2$ holds in *F*(*r*,2), we have

$$\begin{aligned} x_w = x_{w-2}{x_{w-1}}^2 &= {a_2}^{P_{w-2}+P_{w-3}}({a_1}^2)^{P_w-P_{w-1}} = {a_2}^{P_{w-2}+P_{w-3}}({a_2}^2)^{P_w-P_{w-1}} \\ &= {a_2}^{P_{w-2}+P_{w-3}+2P_w-2P_{w-1}} = {a_2}^{2P_{w-2}+P_{w-3}-P_{w-2}+2P_w-2P_{w-1}} \\ &= {a_2}^{P_{w-1}-P_{w-2}+2P_w-2P_{w-1}} = {a_2}^{P_{w-1}+2P_w-2P_{w-1}-P_{w-2}} = {a_2}^{P_w+P_{w-1}}. \end{aligned} \tag{5}$$

2) Let $w \equiv 0\ (mod\ 2)$. Then $x_{w-2} = a_1^{P_{w-1}-P_{w-2}},\ x_{w-1} = a_2^{P_{w-1}+P_{w-2}}$.

Now by definition of x_w, we get

$$\begin{aligned} x_w = x_{w-2}{x_{w-1}}^2 &= {a_1}^{P_{w-1}-P_{w-2}}({a_2}^2)^{P_{w-1}-P_{w-2}} = {a_1}^{P_{w-1}-P_{w-2}}({a_1}^2)^{P_{w-1}-P_{w-2}} \\ &= {a_1}^{3P_{w-1}+P_{w-2}} = {a_1}^{P_w+P_{w-1}} = {a_1}^{2P_w+P_{w-1}-P_w} = {a_1}^{P_{w+1}-P_w}. \end{aligned} \tag{6}$$

Thus the assertion holds.

Theorem 2.2. If *w* is odd number then $P_w + P_{w-1}$ is odd number.

Proof. Use an induction method on *w*. We have for *w*=1; $P_w + P_{w-1} = 1$. Then

$$P_w + P_{w-1} = 2P_{w-1} + P_{w-2} + 2P_{w-2} + P_{w-3} = 2(P_{w-1} + P_{w-2}) + P_{w-2} + P_{w-3}. \tag{7}$$

Then by the hypothesis of induction, the assertion holds.

Theorem 2.3. let *G*=*F*(*r*,2), then per $Q_2(G.X) = hP_2(r-1)$.

Proof. By the above lemma *w*=per $Q_2(G.X)$ even. Then we have

$$\begin{cases} x_w = a_1^{P_{w+1}-P_w} = a_1 \\ x_{w+1} = a_2^{P_{w+1}+P_w} = a_2. \end{cases} \tag{8}$$

Thus we get,

$$\begin{cases} P_{w+1} - P_w \equiv 1\ (mod\ 2\ (r-1)) \\ P_{w+1} + P_w \equiv 1\ \big(mod\ 2\ (r-1)\big). \end{cases} \tag{9}$$

So that,

$$\begin{cases} 2P_w \equiv 0\ \ (mod\ 2\ (r-1)) \\ 2P_{w+1} \equiv 2\ \big(mod\ 2\ (r-1)\big) \end{cases} \tag{10}$$

and,

$$\begin{cases} P_w \equiv 0 \ \ (mod\ (r-1)) \\ P_{w+1} \equiv 1 \left(mod\ (r-1)\right). \end{cases} \tag{11}$$

Then,

$$w{=}h\ P_2(r-1). \tag{12}$$

Thus the result holds.

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