Lie Groups of Symmetries for Nonlinear Ordinary Differential Equations

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Abstract

Differential equations, in particular nonlinear equations, are widely used in formulating many mathematical models and physical problems. However, a few nonlinear differential equations can be solved explicitly. The main purpose of this work is to present a method to obtain exact solutions to first-order Nonlinear Ordinary Differential Equations (NODEs) using Lie symmetry groups. These symmetry groups of NODEs are discussed. Symmetry groups are used to find appropriate change of variables that convert a given first-order NODE into separable form.

Keywords: differential equations, nonlinear, Lie symmetry groups

1 Introduction

There are many different standard techniques to obtain exact solutions of Ordinary Differential Equations (ODEs). But, most of these methods can be applied only for particular types of ODEs such as separable,homogeneous,exact..etc. Marius Sophus Lie who was a Norwegian mathematician discovered that most of these solution techniques are based on the group of continuous symmetries of ODEs[1] and introduced the notion of continuous groups, which is called Lie groups in order to obtain exact solutions to ODEs[7]. Solution methods under Lie groups are called Lie symmetry methods and the importance of these methods is that we are able to apply Lie symmetry methods to ODEs that do not fit into standard types that are considered above.

Nonlinear Ordinary Differential Equations (NODEs) are widely used in formulating many mathematical models and physical problems. But, a few NODEs can be solved explicitly [2]. The most of the standard solution methods are insufficient to obtain exact solutions to NODEs [6]. In this paper, we present basic theories and definitions of continuous symmetry groups of ODEs called Lie groups of symmetries and use Lie groups of symmetries to obtain exact solutions to first order NODEs. Mainly, the Lie groups of symmetries of a given first-order NODE can be applied to obtain appropriate change of variables that can be used to convert the NODE into separable form.

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2 Symmetry groups of ODEs.

To understand the symmetries of ODEs it is helpful to understand the symmetries of planer objects[3]. Consider clockwise rotations of the angles $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ and 2π about the center of equilateral triangle. Since these transformations leave the object apparently unchanged these transformations can be considered as symmetries of the equilateral triangle and flips about three axes also can be considered as symmetries[3]. Hence this triangle has six different symmetries. These symmetries are considered as **discrete symmetries** since they can't be represented by a continuous parameter[3]. Symmetries are called **continuous symmetries** if it can be represented by at least one parameter which can be continuously varied. In this paper we restrict our attention to continuous and smooth symmetries.

Definition 2.1. A Transformation can be considered as smooth symmetry if it satisfies following three conditions[3].

1. The transformation preserves the structure of the object.
2. Transformation is a diffeomorphism
3. Symmetry condition of the object should be satisfied by the transformation.

In this paper we find symmetries for family of solution curves of a given ODE. Therefore rigid curves can be considered as the structure of the objects. Bijective and Infinitely differentiable mapping with infinitely differentiable inverse is called **diffeomorphism**. Since any symmetry has a unique inverse that is also a symmetry[3] smooth symmetry is a diffeomorpism. A diffeomorphism exits between two open subsets of $\mathbb{R}^2$ does not change the properties of objects defined on its domain[1]. A mathematical object and its image under the summery are indistinguishable. That phenomena is called **symmetry condition** of the object [3]. Symmetries of differential equations will be discussed under the next definition interpreted from [7].

Definition 2.2. Let $v = (x_1, x_2, ..., x_n)$,$v \in D, D \subset \mathbb{R}^n$.The set of transformations

$$P_\varepsilon : v \to v^* = \psi(v, \varepsilon)$$

defined for each $v \in D$ depending on parameter $\varepsilon \in S$,$S \subset \mathbb{R}$,forms a one parameter Lie group of transformations on D if it satisfies following conditions. $\phi(\varepsilon, \delta)$ is the law of composition of parameters ε, $\delta \in S$

1. $\forall \varepsilon; \varepsilon \in S$ the transformations are bijections in D, in particular $v^* \in D$.
2. S and ϕ form a group G and $\varepsilon = 0$ corresponds to the identity element e of G
3. ε is a continuous parameter, i.e $S \subset \mathbb{R}$.
4. $v^* = v$ when $\varepsilon =$ e , i.e
$$\psi(v, e) = v$$
5. If $v^* = \psi(v, \varepsilon)$ and $v^{**} = \psi(v^*, \delta)$ then
$$v^{**} = \psi(\psi(v, \varepsilon), \delta) = \psi(v, \phi(\varepsilon, \delta))$$
6. ψ is infinitely differentible with respect to $v \in D$ and analytic function of $\varepsilon \in S$.
7. $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ.

In above definition v^* represents the transformed point $(x_1^*, x_2^*, ..., x_n^*)$. According to the definition 2.2 these point transformations depend on one continuous parameter ε. The group formed by set s and ϕ is called **Local Lie group**[1]. The group of transformations is defined based on this Local Lie group and composition of transformations is the group operation. In this paper composition of two transformations is denoted by o . Next we present how these group of transformations satisfy all group axioms. Notice that

$P_\delta o P_\varepsilon = P_{\phi(\varepsilon,\delta)}$ corresponds to $\psi(\psi(v,\varepsilon),\delta) = \psi(v,\phi(\varepsilon,\delta))$.
Let $\varepsilon, \delta \in S$. Then

$$P_\varepsilon o P_\delta = P_{\phi(\delta,\varepsilon)}$$

By condition 2, $\phi(\delta,\varepsilon) \in S$.Therefore $P_{\phi(\delta,\varepsilon)}$ is also a transformation of the set of transformations. Hence **Closure property** is satisfied.
Let $\varepsilon, \delta, \alpha \in S$. By condition 2, $\phi(\phi(\delta,\varepsilon),\alpha) = \phi(\delta,\phi(\varepsilon,\alpha))$.Then

$$P_\alpha o(P_\varepsilon o P_\delta) = P_{\phi(\phi(\delta,\varepsilon),\alpha)} = P_{\phi(\delta,\phi(\varepsilon,\alpha))} = (P_\alpha o P_\varepsilon) o P_\delta$$

Hence **Associative property** is satisfied.
Let $\varepsilon \in S$. By conditions 2, $\phi(\varepsilon, 0) = \varepsilon = \phi(0,\varepsilon)$. Then

$$P_\varepsilon o P_0 = P_{\phi(0,\varepsilon)} = P_\varepsilon = P_{\phi(\varepsilon,0)} = P_0 o P_\varepsilon$$

Hence P_0 is the **Identity transformation**.
Let $\varepsilon \in S$ and $\varepsilon^{-1} \in S$ be the unique inverse element of ε in group G. By condition 2, $\phi(\varepsilon,\varepsilon^{-1}) = \phi(\varepsilon^{-1},\varepsilon) = 0$.Therefore

$$P_\varepsilon o P_{\varepsilon^{-1}} = P_{\phi(\varepsilon^{-1},\varepsilon)} = P_0 = P_{\phi(\varepsilon,\varepsilon^{-1})} = P_{\varepsilon^{-1}} o P_\varepsilon$$

Therefore for every transformation P_ε there exists **unique inverse** $P_{\varepsilon^{-1}}$. Since ε is a continuous parameter there are infinitely many elements in this group of transformations. Using conditions $2, 3, 4$ and 5 we can obtain following result.

$$\begin{aligned}\psi(v^*,\varepsilon^{-1}) &= \psi(\psi(v,\varepsilon),\varepsilon^{-1}) \\ &= \psi(v,\phi(\varepsilon,\varepsilon^{-1})) \\ &= \psi(v,0) \\ &= v\end{aligned}$$

Therefore inverse transformation of $v^* = \psi(v,\varepsilon)$ is given by $\psi(v^*,\varepsilon^{-1})$. According to condition 6 $\psi(v,\varepsilon)$ is infinitely differentiable with respect to v. It is a differentiation in higher dimension. Since $\psi(v,\varepsilon)$ is infinitely differentiable with respect to v , $\psi(v^*,\varepsilon^{-1})$ is infinitely differentiable with respect to v^*. Hence $\psi(v,\varepsilon)$ is a diffeomorphism. Therefore one parameter Lie group of transformations can be considerd as a diffeomorphism group. In this paper following notations are used to represent One parameter Lie group of transformations acting on space $\mathbb{R}^2$.

$$v^* = (x^*, y^*) = \psi(v,\varepsilon) = (X(x,y;\varepsilon), Y(x,y;\varepsilon)) \tag{1}$$

where $v = (x, y)$, X and Y represents smooth functions and ε is a real parameter.

One parameter Lie group of transformations can be considered as symmetry group of a ODE if the transformations map any solution curve into another solution curve [1]. A Lie group of transformations is said to be **admitted** by a ODE if it is a symmetry group of that ODE [7].

Consider a one parameter Lie group of transformations admitted by a ODE. Then this group of transformations map any solution curve of the ODE into another solution curve. Therefore family of solution curves of the ODE and its image under the transformations are indistinguishable.Then that one parameter Lie group of transformations satisfies the symmetry condition of family of solution curves called **symmetry condition of the ODE**. Mathematical expression for the symmetry condition of ODEs is given in next definition interpreted based on [3] .

Definition 2.3. Consider the ordinary differential equation given by $\frac{d^n y}{dx^n} = f(x, y, \underset{1}{y}, \underset{2}{y}, ..., \underset{n-1}{y})$. Let $v^* = \psi(v,\varepsilon)$ be a one parameter Lie group of transformations that admitted by $f(x, y, \underset{1}{y}, \underset{2}{y}, ..., \underset{n-1}{y})$ where $v^* = (x^*, y^*), v = (x, y)$ and ε is a real parameter. Then the symmetry condition of $f(x, y, \underset{1}{y}, \underset{2}{y}, ..., \underset{n-1}{y})$ is given by

$$\frac{d^n y^*}{dx^{*n}} = f(x^*, y^*, \underset{1}{y^*}, \underset{2}{y^*}, ..., \underset{n-1}{y^*}) \;\; when \;\; \frac{d^n y}{dx^n} = f(x, y, \underset{1}{y}, \underset{2}{y}, .., \underset{n-1}{y}). \tag{2}$$

where $\underset{k}{y} = \frac{d^k y}{dx^k}$ and $\underset{k}{y^*} = \frac{d^k y^*}{dx^{*k}}$, $k = 1, 2, ..., n$

The symmetry condition of a ODE (2) implies that solution curves in (x, y)-plane and its image in (x^*, y^*)-plane are indistinguishable. Hence Lie group symmetry admitted by a ODE map its family of solution curves to itself .Therefore obviously transformations preserve the structure(rigid curves) of family of solution curves. Hence one parameter Lie group of transformations admitted by a ODE satisfy all three conditions in definition 2.1 and this group of transformations is called (**Lie group of symmetries**). Following example illustrates how one parameter Lie group of transformations admitted by a ODE satisfy the symmetry condition of the ODE.

Example 2.4. For this example a differential equation is taken from exercises of [5]. Consider the non linear first order differential equation $\frac{dy}{dx} = 2y^2 + xy^3$. It admits one parameter Lie group of transformation given by

$$v^* = \psi(v, \varepsilon) = (e^{\varepsilon}x, e^{-\varepsilon}y)$$

where $v^* = (x^*, y^*), v = (x, y)$ and ε is a real parameter. We can check whether this is a symmetry group or not for the given ODE by checking the symmetry condition of the ODE. By the chain rule.

$$\begin{aligned}\frac{dy^*}{dx^*} &= \frac{dy^*}{dx} / \frac{dx^*}{dx} \\ &= \frac{e^{-\varepsilon}y'}{e^{\varepsilon}} \\ &= e^{-2\varepsilon}\frac{dy}{dx}\end{aligned}$$

Then we can obtain $2y^{*2} + x^*y^{*3}$ in terms of x and y by substituting the transformations.

$$\begin{aligned}2y^{*2} + x^*y^{*3} &= 2.e^{-2\varepsilon}y^2 + e^{\varepsilon}xe^{-3\varepsilon}y^3 \\ &= 2.e^{-2\varepsilon}(y^2 + xy^3) \\ &= e^{-2\varepsilon}\frac{dy}{dx}\end{aligned}$$

Therefore we can obtain

$$\frac{dy^*}{dx^*} = 2y^{*2} + x^*y^{*3}$$

Therefore $\frac{dy^*}{dx^*} = 2y^{*2} + x^*y^{*3}$ when $\frac{dy}{dx} = 2y^2 + xy^3$. Hence $v^* = \psi(v, \varepsilon)$ satisfies the symmetry condition of given ODE.

Definition 2.5. By expanding v^* about ε=0 in Taylor series we get

$$v^* = v + \varepsilon(\frac{\partial\psi(v,\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}) + \frac{\varepsilon^2}{2}(\frac{\partial^2\psi(v,\varepsilon)}{\partial\varepsilon^2}\bigg|_{\varepsilon=0}) + ...$$

$$v^* = v + \varepsilon(\frac{\partial\psi(v,\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}) + O(\varepsilon^2)$$

$$\vartheta(v) = \frac{\partial\psi(v,\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0} \tag{3}$$

$\vartheta(v)$ are called the **infinitesimals** of Lie group of transformations and following equation is called **infinitesimal generator** of Lie group of transformations

$$\mathbf{X} = \mathbf{X}(v) = \sum_{i=1}^{n} \vartheta_i(v)\frac{\partial}{\partial v_i} \tag{4}$$

The infinitesimals of One parameter Lie group of transformations acting on $\mathbb{R}^2$ space 1 are represented by the standard notations $\xi(x, y)$ and $\eta(x, y)$ where

$$\vartheta(v) = (\frac{dx^*}{d\varepsilon}|_{\varepsilon=0}, \frac{dy^*}{d\varepsilon}|_{\varepsilon=0}) = (\xi(x, y), \eta(x, y))$$

Then the infinitesimal generator becomes

$$\mathbf{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

$(\xi(x,y),\eta(x,y))$ represents tangent vector field of the corresponding one parameter Lie group of transformations[3].

2.1 Linearized Symmetry Condition of First Oder ODEs.

Under this subsection we present how to obtain infinitesimals of Lie group symmetries that admitted by a given ODE.

Consider the first order ordinary differential equation given by $\frac{dy}{dx} = f(x,y)$. Let $v^* = \psi(v,\varepsilon)$ be a one parameter Lie group of transformations that admitted by $f(x,y)$ where $v^* = (x^*,y^*), v = (x,y)$ and ε is a real parameter. Let $x^* = X(x,y;\varepsilon)$ and $y^* = Y(x,y;\varepsilon)$ where X and Y are two functions of x,y and ε. According to the symmetry condition (2)

$\frac{dy^*}{dx^*} = f(x^*,y^*)$ when $\frac{dy}{dx} = f(x,y)$

Then by total derivative operator

$$\frac{D_y y^*}{D_x x^*} = \frac{y^*_x + y' y^*_y}{x^*_x + y' x^*_y} = f(x^*,y^*) \tag{5}$$

when $\frac{dy}{dx} = f(x,y)$. By expanding x^*, y^* and $f(x^*,y^*)$ in Taylor series about $\varepsilon = 0$ we can obtain

$$\begin{aligned} x^* &= X(x,y;\varepsilon) = x + \varepsilon\xi(x,y) + O(\varepsilon^2) \\ y^* &= Y(x,y;\varepsilon) = y + \varepsilon\eta(x,y) + O(\varepsilon^2) \\ f(x^*,y^*) &= f(x,y) + \varepsilon(f_x(x,y)\xi(x,y) + f_y(x,y)\eta(x,y)) + O(\varepsilon^2) \end{aligned} \tag{6}$$

Then by substituting equations 6 into 5 and ignoring terms of order ε^2 and higher.

$$\frac{f + \varepsilon(\eta_x + f\eta_y)}{1 + \varepsilon(\xi_x + f\xi_y)} = f + \varepsilon(f_x\xi + f_y\eta)$$

Then by simplifying we can obtain

$$\eta_x - \xi_y f^2 + (\eta_y - \eta_x)f = \xi f_x + \eta f_y \tag{7}$$

By solving 7 with given ODE we can obtain general solutions for the infinitesimals of Lie symmetries that admitted by the given ODE.The equation 7 is called **Linearized symmetry condition** of first order ODEs. Following example illustrates how to find infinitesimals of Lie group of symmetries admitted by a given ODE using this Linearized symmetry condition.

Example 2.6. Consider the non linear ordinary differential equation $\frac{dy}{dx} = f(x,y) = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$ Then the Linearized symmetry condition 7 becomes

$$\eta_x - \xi_y(xy^2 - \frac{2y}{x} - \frac{1}{x^3})^2 + (\eta_y - \eta_x)(xy^2 - \frac{2y}{x} - \frac{1}{x^3}) = \xi(y^2 + \frac{2y}{x^2} + \frac{1}{x^4}) + \eta(2yx - \frac{2}{x})$$

To solve the Linearized symmetry condition an assumption should be made about the form of the infinitesimals. Assume that $\xi(x,y) = A(x)$ and $\eta(x,y) = B(x)y$ where A and B are functions of x. Then Linerized symmetry condition becomes

$$B'y + (B - A')(xy^2 - \frac{2y}{x} - \frac{1}{x^3}) = A(y^2 + \frac{2y}{x^2} + \frac{3}{x^4}) + (B)(y)(2yx - \frac{2}{x})$$

Then by taking all the terms into R.H.S and multiplying by x^4 we can obtain

$$'yx^4 + (B - A')(x^5y^2 - 2x^3y - x) - A(x^4y^2 + 2x^2y + 3) - B(2x^5y^2 - 2x^3y) = 0 \tag{8}$$

Let $y = 0$, then above equation becomes $(B - A')(-x) - 3A = 0$.

$$B = A' - \frac{3A}{x} \tag{9}$$

Then by comparing coefficients of y^2 in R.H.S and L.H.S of 11 we can obtain

$$(B - A')(x^5) - A(x^4) - B(2x^5) = 0$$

$$(-B - A')x - A = 0$$

By substituting 9 ,

$$(-2A'x + 3A) - A = 0$$

$$A' = \frac{A}{x}$$

By solving above ODE we can obtain

$$\xi(x, y) = A(x) = cx, c \in \mathbb{R}$$

By substituting $A(x) = cx$ to equation 9

$$B = c - \frac{3cx}{x} = -2c$$

Then $\eta(x, y) = B(x)y = -2cy$.

Notice that if a ODE admits one parameter Lie group of transformations given by

$$(x^*, y^*) = (x, y + \varepsilon) \tag{10}$$

then one solution curve of the ODE is mapped to any other solution curves into the direction of y without changing the variable x. It implies that the ODE does not depend on the variable y. Hence it exits in variable separable form.Following justification is interpreted based on [3]

Proof. Let $\frac{dy}{dx} = f(x, y)$ be the given ODE. Then assume that it admits 10 . By the symmetry condition of $\frac{dy}{dx} = f(x, y)$ we can obtain

$$\frac{dy^*}{dx^*} = f(x^*, y^*) = f(x, y + \varepsilon)$$

Using total derivative operator

$$\begin{aligned} \frac{dy^*}{dx^*} &= \frac{y^*_x + y^*_y . y'}{x^*_x + x^*_y . y'} \\ &= \frac{dy}{dx} \\ &= f(x, y) \end{aligned}$$

Therefore $f(x, y + \varepsilon) = f(x, y)$. Hence $f(x, y)$ does not depend on variable s. Then the ODE $\frac{dy}{dx} = f(x, y)$ can be written in the form $\frac{dy}{dx} = f(x)$. Therefore under this change of variable given ODE exits in variable separable form.

□

Let $v^* = \psi(v, \varepsilon)$ be a Lie group of symmetries admitted by a given ODE $f(x, y)$ where $v^* = (x^*, y^*)$ and $v = (x, y)$. Then our aim is to find suitable change of variables that convert $\psi(v, \varepsilon)$ into 10 . Then under this change of variables given ODE exits in variable separable form since it admits $v^* = \psi(v, \varepsilon)$ in terms of new variables. Hence we introduce coordinates

$$(r, s) = (r(x, y), s(x, y))$$

such that

$$(r^*, s^*) = (r(x^*, y^*), s(x^*, y^*)) = (r, s + \varepsilon) \tag{11}$$

[3].

According to transformations 11

$$\left.\frac{dr^*}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

By chain rule

$$\xi(x, y)r_x + \eta(x, y)r_y = 0 \tag{12}$$

Similarly we can obtain

$$\xi(x, y)s_x + \eta(x, y)s_y = 1 \tag{13}$$

Hence

$$\mathbf{X}r = 0 \qquad \mathbf{X}s = 1$$

where $\mathbf{X}$ is the infinitesimal generator of v^*. And these coordinates should be invariable. Hence these coordinates should satisfy following property

$$r_x s_y - r_y s_x \neq 0 \tag{14}$$

Coordinates that satisfies 12,13 and 14 are called **Canonical coordinates** of corresponding one parameter Lie group of transformations[3]. By solving the partial differential equations 12 and 13 using method of characteristics we can obtain the coordinates r and s. Following two examples illustrate how to obtain exact solutions of NODEs by converting them into variable separable form using these canonical coordinates of its admitted Lie group of symmetries.

Example 2.7. For this example a differential equation is taken from [3] to find general solution using Lie symmetries and illustrate invariant curves.
Consider the Ricaati type first order non linear ordinary differential equation

$$\frac{dy}{dx} = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$$

In example 2.6 we obtained general solutions for the infinitesimals of One parameter Lie group of transformations that admitted by above ODE. $\xi(x, y) = cx$ and $\eta(x, y) = -2cy$. Let $c = 1$.Then the infinitesimals become $(x, -2y\ )$. Then the infinitesimal generator is given by

$$\mathbf{X} = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$$

Let (r, s) be canonical coordinates of the One parameter Lie group of transformations. Then the equations 12 and 13 become

$$x\frac{\partial r}{\partial x} - 2y\frac{\partial r}{\partial y} = 0$$

$$x\frac{\partial s}{\partial x} - 2y\frac{\partial s}{\partial y} = 1$$

Then we can obtain the independent coordinate r by solving the characteristic equation $\frac{dx}{x} = \frac{dy}{-2y}$.

$$\int \frac{dx}{x} = \int \frac{dy}{-2y}$$
$$y = cx^{-2}$$
$$c = x^2 y, \quad c \in \mathbb{R}$$

Therefore $r(x,y) = c = x^2 y$. we can find the dependent coordinate s by solving the characteristic equation $\frac{dx}{x} = \frac{dy}{-2y} = \frac{ds}{1}$.

$$s(x,y) = \int \frac{ds}{1} = \int \frac{dx}{x}$$
$$s(x,y) = ln(x) + c \qquad ,c \in \mathbb{R}$$

Hence the canonical coordinates are given by $(r(x,y), s(x,y) = (x^2 y, ln(x))$.Then we can convert the ODE into variable separable form using these canonical coordinates. By total derivative operator

$$\begin{aligned} \frac{ds}{dr} &= \frac{s_x + y' s_y}{r_x + y' r_y} \\ &= \frac{\frac{1}{x}}{2xy + (xy^2 - \frac{2y}{x} - \frac{1}{x^3})x^2} \\ &= \frac{1}{((x^2y)^2 - 1)} \\ &= \frac{1}{(r^2 - 1)} \end{aligned}$$

Therefore under the canonical coordinates the given ODE exists in a variable separable form.Therefore we can obtain the solution by integration.

$$\int ds = \int \frac{dr}{(r^2 - 1)}$$
$$s = \frac{1}{2} ln(\frac{r-1}{r+1}) + c \quad ,c \in \mathbb{R}$$

Then we can get the general solution of the given ODE by changing the canonical coordinates to original coordinates.

$$y = \frac{c_1 + x^2}{x^2(c_1 - x^2)} \qquad ,c_1 \in \mathbb{R}$$

Example 2.8. For this example a differential equation is taken from exercises of [5] to find general solution using Lie symmetries and illustrate invariant curves.
Consider the first order non linear ordinary differential equation

$$\frac{dy}{dx} = \frac{x^3}{y} - xy$$

.It admits a one parameter Lie group of transformations whose tangent vector field is $(\xi(x,y), \eta(x,y)) = (\frac{1}{x}, \frac{1}{y})$.we cam obtain these infinitesimals by solving the linerized symmetry condition 7. Then the infinitesimal generator is given by

$$\mathbf{X} = \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{y}\frac{\partial}{\partial y}$$

Let (r,s) be canonical coordinates of the One parameter Lie group of transformations. Then the equations 12 and 13 become

$$\frac{1}{x}\frac{\partial r}{\partial x} + \frac{1}{y}\frac{\partial r}{\partial y} = 0$$

$$\frac{1}{x}\frac{\partial s}{\partial x} + \frac{1}{y}\frac{\partial s}{\partial y} = 1$$

Then we can find independent coordinate r by solving the characteristic equation $\frac{dx}{x^{-1}} = \frac{dy}{y^{-1}}$.

$$\int xdx = \int ydy$$
$$x^2 + c = y^2 \quad , c \in \mathbb{R}$$
$$c = -x^2 + y^2$$

Then $r(x, y) = c = -x^2 + y^2$
we can find the dependent coordinate s by solving the characteristic equation $\frac{dx}{x^{-1}} = \frac{dy}{y^{-1}} = ds$.

$$xdx = ydy = ds$$
$$s = \int ds = \int xdx$$
$$s = \frac{x^2}{2} + c \qquad c \in \mathbb{R}$$

Hence the canonical coordinates are given by $(r(x, y), s(x, y)) = (-x^2 + y^2, \frac{x^2}{2})$. Then we can convert the ODE into variable separable form using these canonical coordinates. By total derivative operator

$$\begin{aligned}\frac{ds}{dr} &= \frac{s_x + y's_y}{r_x + y'r_y}\\ &= \frac{x}{-2x + (\frac{x^3}{y} - xy)2y}\\ &= \frac{-1}{2(-x^2 + y^2 + 1)}\\ &= \frac{-1}{2(r + 1)}\end{aligned}$$

Therefore under the canonical coordinates the given ODE exists in a variable separable form.Therefore we can obtain the solution by integration.

$$\int ds = \int \frac{-1}{2(r + 1)dr}$$
$$s = \frac{-1}{2} ln(r + 1) + c_1 \quad , c_1 \in \mathbb{R}$$

Then we can get the general solution of the given ODE by changing the canonical coordinates into original coordinates.

$$y = \pm\sqrt{-1 + x^2 + \frac{c_2}{e^{x^2}}} \qquad , c_2 \in \mathbb{R}$$

3 Conclusions

In this paper, fundamental theorems and definitions regarding Lie symmetry methods have been presented. The main purpose of this paper is to present how to obtain exact solutions to first order NODEs, using Lie groups of symmetries since most of standard solution techniques are insufficient to get exact solutions to NODEs.

First we discussed about symmetry groups of ODEs using One parameter Lie group of transformations. In examples 2.7 and 2.8, we obtained general solutions to a couple of first-order NODEs by converting them into variable separable form using Lie groups of symmetries. In the all theorems and definitions that we presented above the standard types of ODEs were not taken into consideration. Hence we can apply this method to find exact solutions unfamiliar types of ODEs but for some ODEs solving the Linearized symmetry condition by hand can be very hard and complex. But there are many software including Mathematica and Maple which have packages that can be used to do calculations of Lie symmetry analysis. In such cases we can use these software packages to solve the Linearized symmetry condition.

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