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Solvable Graph of finite Group

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Abstract

Let G be a finite non-solvable group with solvable radical $Sol(G)$. The solvable graph $\Gamma_{sol}(G)$ of group G is a graph with vertex set $V(\Gamma_{sol}) = \{\sigma \mid \sigma \in G\}$ and two distinct vertices σ_1 and σ_2 are adjacent if and only if $\langle \sigma_1, \sigma_2 \rangle$ is solvable group, so the solvability degree of G is define by the number of all elements such that $\{(\sigma_1, \sigma_2) \in G \times G \mid \langle \sigma_1, \sigma_2 \rangle \leq_{Sol} G\}$ on the number $(G)^2$. We show that the relation between $\Gamma_{sol}(G)$ and the solvability degree of G.

Keywords: Solvable group, Solvable graph, Solvability degree
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1 Introduction

Let $\Gamma(V, E)$ be a simple graph. The set of vertices denoted by $V(\Gamma)$ and the set of edges denoted by $E(\Gamma)$.

The solvable Graph of a finite group G denoted by $\Gamma_{sol}(G)$ was introduced by Ma et. all in [2] in the year 2014. The graph $\Gamma_{sol}(G)$ has vertex set as elements of the non-solvable group G and any two vertices σ_i and σ_j are adjacent in $\Gamma_{sol}(G)$ if and only if $\langle \sigma_i, \sigma_j \rangle \leq_{Sol}$ is solvable subgroup of G. In this paper we take the generalizer of non-solvable group of type $C_p \times A_5$ it is will-known the A_5 is smallest non-solvable group, thus $C_p \times A_5$ is non-solvable group. It is clear that if group G is a solvable, then $\Gamma_{sol}(G) \cong K_{|G|}$ since for any two elements a, b of G the subgroup $\langle a, b \rangle$ is solvable in G.

In this paper, we consider a simple graph which is undirected, with no loops or multiple edges. Let Γ be a graph. We will denote by $V(\Gamma)$ and $E(\Gamma)$, the set of vertices and edges of Γ, respectively. The degree of a vertex $v \in V(\Gamma)$ is denoted by $deg(v)$, and it well-known that $deg(v) = |N(v)|$. The degree sequence of a graph with vertices $v_1, \cdots, v_n$ is $d = (deg(v_1), \cdots, deg(v_n))$. Every graph with the degree sequence d is a realization of d. A degree sequence is unigraphic if all its realizations are isomorphic. We can present it by $\Delta(\Gamma) = \begin{pmatrix} n_1 & n_2 & \cdots & n_s \\ \mu(n_1) & \mu(n_2) & \cdots & \mu(n_s) \end{pmatrix}$, where n_i are degree vertices and $\mu(n_i)$ are multiplicities. The split graph is a graph in which the vertices can be partitioned into a clique and an independent set.

Suppose that g an element of group G, the solvabilizer of g define by $\{y \in G \mid \langle g, y \rangle\} \leq_{sol}$ in G and denoted by $Sol_G(g)$ and the centralizer of g is given by $Cent_G(g) = \{y \in G \mid gy = yg\}$ where $Cent_G(g) \subset Sol_G(g)$ and $|Sol_G(g)|$ divided $Cent_G(g)$ for each $g \in G$ for more see [1, 2]. It is clear that is not necessarily a subgroup of G. It is easy to see that $Sol(G) = \{(u, v) \in G \times G, \langle u, v \rangle \leq_{sol} G\} = \bigcup_{\forall u \in G} Sol_G(u)$. Also, $Sol(G)$ is the solvable radical of G (see [3]).

[1] speaker

Let G be a finite group and non-solvable, the probability that a randomly chosen pair of elements of G generate a solvable group is define by:

$$P_{sol}(G) = \frac{|\{(g,y) \in G \times G \mid \langle g,y \rangle \leq_{sol} G\}|}{|G|^2}$$

$P_{sol}(G)$ is the probability that a randomly chosen pair of elements of G generate a solvable group (see [4, 5]).

we can present conjugate definition by from using the conjugacy class $Cl_G(g)$,

$$\begin{aligned} |Sol(G)| &= |\{(u,v) \in G \times G \mid \langle u,v \rangle \leq_{sol} G\}| \\ &= \bigcup_{\forall u \in G} |\{v \in G \mid \langle u,v \rangle \leq_{sol} G\}| \\ &= \sum |cl_G(u)||sol_G(u)| \end{aligned}$$

We introduce in this paper some of important relation between solvable graph $\Gamma_{sol}(G)$ of G and probability that a randomly chosen pair of elements of G generate a solvable group $P_{sol}(G)$.

2 Main results

Proposition 2.1. *: Let $G \cong A_5$, the solvability degree of elements is given by:*

1. $Sol_{A_5}(e) = \{g \mid \forall g \in A_5\}$;

2. $Sol_{A_5}((ab)(cd)) = \begin{cases} g & \#(g) \\ Identity & 1 \\ (bc)(de),(bd)(ce),(be)(cd),(ab)(ce), & \\ (ab)(cd),(ab)(ce),(ac)(de),(ac)(bd), & \\ (ac)(be),(ad)(ce),(ad)(bc),(ad)(be), & \\ (ae)(cd),(ae)(bc),(ae)(bd) & 15 \\ (abc)^{\pm},(abd)^{\pm},(abe)^{\pm},(cda)^{\pm},(cdb)^{\pm},(cde)^{\pm} & 12 \\ (abced)^{\pm},(abdec)^{\pm},(acdbe)^{\pm},(aebcd)^{\pm} & 8 \end{cases}$,

3. $Sol_{A_5}((abc)) = \begin{cases} g & \#(g) \\ Identity & 1 \\ (ab)(ij)_{i<j,i\neq j\neq a,b},(ac)(ij)_{i<j,i\neq j\neq a,b}(bc)(ij)_{i<j,i\neq j\neq a,b} & 9 \\ (abi)^{\pm}_{i=c,d,e},(acj)^{\pm}_{j=b,e},(bcj)^{\pm}_{j=d,e} & 14 \end{cases}$,

4. $Sol_{A_5}((abcde)) = \begin{cases} g & \#(g) \\ Identity & 1 \\ (be)(cd),(ab)(ce),(ac)(de),(ad)(bc),(ae)(bd) & 5 \\ (abcde),(acebd),(adbec),(aedcb) & 4 \end{cases}$,

5. $Sol_{A_5}((abced)) = \begin{cases} g & \#(g) \\ Identity & 1 \\ (bd)(ce),(ab)(cd),(ac)(de),(ad)(bc),(ae)(bc) & 5 \\ (abced),(acdbe),(adecb),(aebdc) & 4 \end{cases}$.

Corollary 2.2. *1. If $g = e$, then $|Sol_{A_5}(e)| = 60$*

2. If $g = (ab)(cd)$, then $|Sol_{A_5}((ab)(cd))| = 36$

3. *If* $g = (abc)$*, then* $|Sol_{A_5}((abc))| = 24$

4. *If* $g = (abcde)$*, then* $|Sol_{A_5}((abcde))| = 10$

5. *If* $g = (abced)$*, then* $|Sol_{A_5}((abced))| = 10$

Proposition 2.3. *:*
$Con_{A_5}(e) = \{g \mid \forall g \in A_5\}$

$$Con_{A_5}((ab)(cd)) = \begin{cases} g & \#(g) \\ (bc)(de),(bd)(ce),(be)(cd),(ab)(ce), & \\ (ab)(cd),(ab)(ce),(ac)(de),(ac)(bd), & \\ (ac)(be),(ad)(ce),(ad)(bc),(ad)(be), & \\ (ae)(cd),(ae)(bc),(ae)(bd) & 15 \end{cases},$$

$$Con_{A_5}((abc)) = \begin{cases} g & \#(g) \\ (cde),(ced),(bcd),(bce),(bdc),(bde),(bec),(bed), & \\ (abc),(abd),(abe),(acb),(acd),(ace),(adb),(adc), & \\ (ade),(aeb),(aec),(aed) & 20 \end{cases},$$

$$Con_{A_5}((abcde)) = \begin{cases} g & \#(g) \\ (abcde),(abdec),(abecd),(acedb),(acbed),(acdbe), & \\ (adceb),(adebc),(adbce),(aedcb),(aebdc),(aecbd) & 12 \end{cases},$$

$$Con_{A_5}((abced)) = \begin{cases} g & \#(g) \\ (abced),(abdce),(abedc),(acdeb),(acbde),(acebd), & \\ (adecb),(adbec),(adcbe),(aecdb),(aedbc),(aebcd) & 12 \end{cases}.$$

Corollary 2.4. *The following held:*

1. *If* $g = e$*, then* $|Con_{A_5}(e)| = 1$

2. *If* $g = (ab)(cd)$*, then* $|Con_{A_5}((ab)(cd))| = 15$

3. *If* $g = (abc)$*, then* $|Con_{A_5}((abc))| = 20$

4. *If* $g = (abcde)$*, then* $|Con_{A_5}((abcde))| = 12$

5. *If* $g = (abced)$*, then* $|Con_{A_5}((abced))| = 12$

Corollary 2.5. *:The results of* A_5*:*

typesofelement	*order*	*ConjugacyClass(y)*	*Size*	*Sol(y)*
C_1	1	()	1	60
C_2	2	$(ab)(cd)$	15	36
C_3	3	(abc)	20	24
C_5	5	$(abcde)$	12	10
D_{10}	10	$(abced)$	12	10

2.1 Parameters of Alternating Group

In this section we will present the generalization formula to compute the number of solvability degree of Alternating group $C_p \times A_5$ where p is an prime number.

We can compute this To compute the parameter of solavable groups of A_5 used ConjugacyClass elements of $A_5 = \{e, (ab)(cd), (abc), (abcde), (abced)\}$ and solvable groups in the rule:

$$P_{sol}(A_5) = \frac{\sum |Con(y)| deg_{sol}(y)}{|G|^2}$$

$$P_{sol}(A_5) = \frac{1(60) + 15(36) + 20(24) + 12(10) + 12(10)}{|60|^2} = \frac{11}{30}$$

see 2.5

A_5	e	(ab)(cd)	(abc)	(abcde)	(abced)
e	1	1	1	1	1
$(ab)(cd)$	1	1	a_{ij}	b_{ij}	c_{ij}
(abc)	1	d_{ij}	e_{ij}	0	0
$(abcde)$	1	f_{ij}	0	g_{ij}	h_{ij}
$(abced)$	1	k_{ij}	0	n_{ij}	m_{ij}

where

1. $a_{ij} = \begin{cases} 1 & if\ i = (ab)(cd), j = (abc)\ see\ Proposition 2.1 \\ 0 & Otherwise \end{cases}$,

2. $b_{ij} = \begin{cases} 1 & if\ i = (ab)(cd), j = (abcde)\ see Proposition 2.1 \\ 0 & Otherwise \end{cases}$,

3. $c_{ij} = \begin{cases} 1 & if\ i = (ab)(cd), j = (abced) see Proposition 2.1 \\ 0 & Otherwise \end{cases}$,

4. $d_{ij} = \begin{cases} 1 & if\ i = (abc), j = (ab)(cd)\ see\ Proposition 2.1 \\ 0 & Otherwise \end{cases}$,

5. $e_{ij} = \begin{cases} 1 & if\ i = (abc), j = (abc)\ see\ Proposition 2.1 \\ 0 & Otherwise \end{cases}$,

6. $f_{ij} = \begin{cases} 1 & if\ i = (abcde), j = (ab)(cd)\ see\ Proposition 2.1 \\ 0 & Otherwise \end{cases}$,

7. $g_{ij} = \begin{cases} 1 & if\ i = (abcde), j = (abcde)\ see\ Proposition 2.1 \\ 0 & Otherwise \end{cases}$,

8. $h_{ij} = \begin{cases} 1 & if\ i = (abcde), j = (abced)\ see\ Proposition 2.1 \\ 0 & Otherwise \end{cases}$,

9. $k_{ij} = \begin{cases} 1 & if\ i = (abced), j = (ab)(cd)\ see\ Proposition 2.1 \\ 0 & Otherwise \end{cases}$,

10. $n_{ij} = \begin{cases} 1 & if\ i = (abced), j = (abced)\ see\ Proposition 2.1 \\ 0 & Otherwise \end{cases}$,

11. $m_{ij} = \begin{cases} 1 & if\ i = (abced), j = (abced)\ see\ Proposition 2.1 \\ 0 & Otherwise \end{cases}$.

2.2 The relation between $P_{sol}(G)$ and $\Gamma_{sol}(G)$.

We begin with the following Proposition.

Proposition 2.6. *Let $\Gamma_{sol}(G)$ be a simple and solvable graph, The number of degree vertices $def(v)$ for any $v \in V(\Gamma_{sol})$ is equal to $Sol_G(v) - 1$*

Proof. It is clear that the $Sol_G(v) = \{u \in G \mid \langle v, u\rangle \leq_{sol} G\}$ and $deg(v)$ represents the number of vertices from G which are adjacent to v. Since $v \in Sol_G(v)$, therefore $|Sol_G(v)| - 1$ represents the number of vertices which are adjacent to v. Thus $deg(v) = |Sol_G(u)| - 1$. □

Proposition 2.7. *The matrix degree sequences of solvable graph is given by:*

$$\Delta(\Gamma_{sol}(A_5)) = \begin{pmatrix} 1 & 15 & 20 & 24 \\ 59 & 35 & 23 & 9 \end{pmatrix}$$

Proposition 2.8. *The number of edges of solvable graph is given by:*

$$E(\Gamma_{sol}(A_5)) = 630$$

Lemma 2.9. *Let Γ_G be a solvable graph. The following are held:*

1. $deg(v) = |Sol_G(v)| - 1$;
2. $\mu(deg(v)) = |cl_G(v)|$.

Theorem 2.10.

$$P_{sol}(G) = \frac{2|E(\Gamma_{sol}(G))|}{|G|^2}$$

Proof. In the first, the parameters solvablitiy degree is define by $P_{sol}(G) = \dfrac{\{(u, v) \in G \times G, \langle u, v\rangle \leq_{sol} G\}}{|G|^2}$, Let $|G| = n$, suppose that u_i and u_j are elements in G and $cl_G(u_i)$ where $1 \leq i \leq r$, we can used this definition by

$$\begin{aligned} P_{sol}(G) &= \frac{|\{(u_i, u_j) \in G \times G, \langle u_i, u_j\rangle \leq_{sol} G\}|}{|G|^2} \\ &= \frac{|Sol_G(u_1) \cup Sol_G(u_2) \cup \cdots \cup Sol_G(u_n)|}{|G|^2} \\ &= \frac{|Sol_G(u_1)| + |Sol_G(u_2)| + \cdots + |Sol_G(u_n)|}{|G|^2} \\ &= \frac{|cl_g(u_1)||Sol_G(u_1)| + |cl_g(u_2)||Sol_G(u_2)| + \cdots + |cl_g(u_r)||Sol_G(u_r)|}{|G|^2} \\ &= \frac{\sum_{1\leq i\leq r} |cl_g(u_i)||Sol_G(u_i)|}{|G|^2} \\ &= \frac{2|E(\Gamma_{sol}(G))|}{|G|^2}. \end{aligned}$$

□

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