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Relation between Normality degree and Normal Graph of Some of Finite Groups

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Abstract

Let G be a finite group. The normality degree is given by $P_{nd}(G) = \frac{|\{(x,y)\in G\times G|\langle x,y\rangle \trianglelefteq G, \forall x,y\in G\}|}{|G|^2}$ and the normal graph $\Gamma_G(V,E) = \Gamma_N$ is defined by the set of all vertices of $E(\Gamma_N) = \{\{x,y\} \mid \langle x,y\rangle \trianglelefteq G\}$. In this paper, we establish some properties of the normal graph defined by group D_{2n} and study the relation between Γ_N and $P_{nd}(G)$.

Keywords: Normal Graph, Normality Degree, Dihedral Group
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1 Introduction

Let G be a finite group. Computing or study the probability satisfies condition between two elements in a group is the important application of computational group theory. Erfanian in 2013 with outhers introduced some of the probability in finite groups, one of those probability to compute the probability that a randomly chosen pair of elements of G generate a normal subgroup in group is define by:

$$P_{nd}(G) = \frac{|\{(x,y) \in G \times G|\langle x,y\rangle \trianglelefteq G, \forall x,y \in G\}|}{|G|^2}$$

In this paper, we will study and compute the number normality degree elements of the dihedral group $D_{2n} = \langle a,b \mid a^n = b^2 = e, bab = a^{-1}\rangle$ and Dicyclic group $T_{4n} = \langle a,b \mid a^{2n} = b^4 = e, a^n = b^2, bab^{-1} = a^{-1}\rangle$. We denoted that $P_{nd}(G), N_G(x), Sub(G)$ and $N(G)$ the number of normalitiy degree, normalizer of element, subgroups and normal subgroups.

The conjugate element of group G is define for any two elements g and h are conjugate if there exist element $t \in G$ such that $tgt^{-1} = h$ and denoted by $cl(g)$ for the set of all elements are conjugate to g, we can say the subgroup H and K are conjugate if conjugate if there exist element $t \in G$ such that $tHt^{-1} = K$. Recall that the order of group $|G| = |cl(g)||C_G(h)|$, the $C_G(h)$ is centralizer of element g.

The Dihedral group D_{2n} id presented by $D_{2n} = \langle a,b|a^n = b^2 = e, bab = a^{-1}\rangle$ of order $2n$ and generated by two elements a,b, where a of order n and b of order 2. So the dicyclic group of order $4n$ again generated by two elements and presented by $T_{4n} = \langle a,b|a^{2n} = b^2 = e, bab^{-1} = a^{-1}\rangle$.

Cavior in ([2], 1975) and Calhoun ([1], 1987) computed the number of subgroups of the Dihedral group D_{2n} which is equal to $\tau(n) + \sigma(n)$ and presented by :

[1]speaker

1. A subgroups $\langle a^i \rangle$ for all $i \mid n$;
2. A subgroups $\langle a^i, a^j b \rangle$ for all $i \mid n$ and $1 \le j \le i$.

ans Shelash and Ashrafi [4, 5], computed the number of subgroups and presented the structure of subgroups of Dicyclic group T_{4n}.

1. A subgroups $\langle a^i \rangle$ for all $i \mid 2n$;
2. A subgroups $\langle a^i, a^j b \rangle$ for all $i \mid n$ and $1 \le j \le i$.

where $\tau(n)$ is the number of divisors of n and $\sigma(n)$ is the sum of divisors of n. And the structure normal subgroups are define by :

Theorem 1.1. *The following are held:*

1. *If $G \cong D_{2n}$, then the normal of subgroups are:*
   1.1. *A normal subgroups $\langle a^i \rangle$ for all $i \mid n$;*
   2.2. *A normal subgroups $\langle a^i, a^j b \rangle$ for all $i = 1, 2$ and $1 \le j \le i$.*
2. *If $G \cong T_{4n}$, then the normal of subgroups are:*
   2.1. *A normal subgroups $\langle a^i \rangle$ for all $i \mid 2n$;*
   2.2. *A normal subgroups $\langle a^i, a^j b \rangle$ for all $i = 1, 2$ and $1 \le j \le i$.*

In this paper, we consider a simple graph which is undirected, with no loops or multiple edges. Let Γ be a graph. We will denote by $V(\Gamma)$ and $E(\Gamma)$, the set of vertices and edges of Γ, respectively. The degree of a vertex $v \in V(\Gamma)$ is denoted by $deg(v)$, and it well-known that $deg(v) = |N(v)|$. The degree sequence of a graph with vertices $v_1, \cdots, v_n$ is $d = (deg(v_1), \cdots, deg(v_n))$. Every graph with the degree sequence d is a realization of d. A degree sequence is unigraphic if all its realizations are isomorphic. We can present it by $d(\Gamma) = \begin{pmatrix} n_1 & n_2 & \cdots & n_s \\ \mu(n_1) & \mu(n_2) & \cdots & \mu(n_s) \end{pmatrix}$, where n_i are degree vertices and $\mu(n_i)$ are multiplicities. The order of the largest clique in Γ is its clique number and we denoted by $CN(\Gamma)$ In the group theory,

2 Main Results

In this section we will present a new parameters about some of finite groups to study the relationship between those parametrise.

Definition 2.1. Let G be a finite group, the Normality degree of the group G is define by

$$P_{rn}(G) = \frac{|\{(x, y) \in G \times G | \langle x, y \rangle \trianglelefteq G, \forall x, y \in G\}|}{|G|^2}$$

where $\langle x, y \rangle$ is a normal subgroup of G and we denoted by $P_{rn}(G)$.

The value $P_{rn}(G)$ is $0 < P_{rn}(G) \le 1$, we recall that for a finite group G we have $P_{rn}(G) = 1$ if and only if G is an abelain group.

For any element in finite group have a normalizer is define by $N_G(x) = \{y \in G | (x, y) \in G \times G | \langle x, y \rangle \trianglelefteq G\}$ is called the set of elements in G that vote in favor of elements normality is called the normalizer of x in G and denoted $N_G(x)$.

In general, the normalizer $N_G(x)$ of elements is not necessary a subgroup of G. For example, let $G = D_6$, we can see the $N_G((1, 2)) = \{(1, 3), (2, 3), (1, 2, 3), (1, 3, 2)\}$ is not a subgroup in D_6.

Now we may define the subgroup

$N_G(G) = \cap_{\forall x \in G} N_G(x)$

Theorem 2.2. *Let G be a finite group, the Normality degree of the group G is given by:*

$$P_{rn}(G) = \frac{\sum_{\forall x \in G} |N_G(g)|}{|G|^2}$$

Corollary 2.3. *Let G be abelain group, then $P_{rn}(G) = 1$.*

Corollary 2.4. *Let G be a finite group, then*

$$P_{rn}(G) = \frac{\sum_{x \in cl(g)} |N_G(g)|}{C_G(g)|G|}$$

Theorem 2.5. *If G and H are groups, then $P_{rn}(G \times H) = P_{rn}(G) \times P_{rn}(H)$*

Proof. It is clear that $G \times H = \{(g, h) : g \in G \& h \in H\}$,

$$P_{rn}(G \times H) = \frac{\{(g_1h_1, g_2h_2) \in G^2 \times H^2, \langle g_1h_1, g_2h_2 \rangle \trianglelefteq G \times H\}}{(|G||H|)^2}$$

we can take any element of $g_kh_k \in G \times H$ to prove that $\langle g_1h_1, g_2h_2 \rangle$ is a normal subgroup of $G \times H$.

$$\begin{aligned} g_kh_k\langle g_1h_1, g_2h_2 \rangle h_k^{-1}g_k^{-1} &= \langle g_kh_kg_1h_1h_k^{-1}g_k^{-1}, g_kh_kg_2h_2h_k^{-1}g_k^{-1} \rangle \\ &= \langle g_1h_1, g_2h_2 \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} P_{rn}(G \times H) &= \frac{\{(g_1h_1, g_2h_2) \in G^2 \times H^2, \langle g_1h_1, g_2h_2 \rangle \trianglelefteq G \times H\}}{(|G||H|)^2} \\ &= \frac{\{(g_1, g_2) \in G^2 \langle g_1, g_2 \rangle \in G \times (h_1, h_2) \times H^2, \langle h_1, h_2 \rangle \trianglelefteq H\}}{(|G||H|)^2} \\ &= \frac{\{(g_1, g_2) \in G^2 \langle g_1, g_2 \rangle \in G\}}{|G|^2} \times \frac{\{(h_1, h_2) \times H^2, \langle h_1, h_2 \rangle \trianglelefteq H\}}{|H|^2} \\ &= P_{rn}(G) \times P_{rn}(H). \end{aligned}$$

□

Proposition 2.6. *If $H \leq G$, then $P_{rn}(G) \leq P_{rn}(H)$.*

Proof. **Case 1.** If G is a belain group, then by Corollary 2.3 $P_{rn}(G) = P_{rn}(H)$ for all $H \leq G$.

Case 2. Let G be a non-belain group, If $H = \{e\}$, then $P_{rn}(G) \leq P_{rn}(H)$ is true. Otherwise $|H| \leq |G|$ for all subgroups of group G, and $|\{(x, y) \in H \times H, \langle x, y \rangle \trianglelefteq H\}| \leq |\{(x, y) \in G \times G, \langle x, y \rangle \trianglelefteq G\}|$, thus $P_{rn}(G) \leq P_{rn}(H)$. □

In the following lemma we will present some of important properties of normality degree parameter $P_{rn}(G)$ for some of finite group.

Lemma 2.7. *The following are held:*

1. $P_{rn}(D_6) = \frac{3}{4}$ *is a largest value for smallest non-abelain group.*

2. *for* $n \to \inf$, *the* $P_{rn}(D_{2n}) = \frac{1}{4}$

Definition 2.8. Let G be a finite group and x any element in G, the normality degree of elements in group G is define by:

$$\eta(x,y) = \begin{cases} 1 & \text{if } \langle x,y\rangle \trianglelefteq G \;\; \exists y \in G \\ 0 & \text{Otherwise} \end{cases}.$$

and entry in normalitiy degree of elements table is $\eta(x,y)$

η	$\cdots$	y	$\cdots$
$\vdots$		$\vdots$	
x	$\cdots$	$\eta(x,y)$	
$\vdots$			

Lemma 2.9. *Let n be a positive integer number and $G \cong C_n$, the normality degree elements table be isomorphic to matrix of ones $J_{n\times n}$.*

Theorem 2.10. *Let n be an odd positive integer number, the normality degree elements table of the dihedral group D_{2n} is given by the following table:*

η	$\cdots$	a^j	$\cdots$	$\cdots$	a^jb	$\cdots$
$\vdots$						
a^i		$\eta(a^i,a^j)$			$\eta(a^i,a^jb)$	
$\vdots$						
a^ib		$\eta(a^ib,a^j)$			$\eta(a^ib,a^jb)$	
$\vdots$						

the entries is define by:

1. $\eta(a^i,a^j) = 1$ *for all* $1 \le i,j \le n$;
2. $\eta(a^i,a^jb) = \begin{cases} 1 & if \;\; gcd(i,n)=1 \\ 0 & Otheriwse \end{cases}$ *for all* $1 \le i,j \le n$;
3. $\eta(a^ib,a^j) = \eta(a^i,a^jb)$;
4. $\eta(a^ib,a^jb) = \begin{cases} 1 & if \;\; gcd(j-i,n)=1 \\ 0 & Otheriwse \end{cases}$ *for all* $1 \le i,j \le n$.

Proof. For $\eta(a^i,a^j)=1$ it is clear that for any $\langle a^i,a^j\rangle$ is normal subgroup of group D_{2n}.

Since $\langle a^i,a^jb\rangle$ is normal when $i=1$ if n is an odd number, thus $\eta(a^i,a^jb) = \begin{cases} 1 & if \;\; gcd(i,n)=1 \\ 0 & Otheriwse \end{cases}$ is true for any a^i have order n. So $\langle a^ib,a^jb\rangle$ is normal when $gcd(i-j,n)=1$ this mean $a^iba^jb = a^{i-j}$ element of order n, it easy to see that $\langle a^ib,a^jb\rangle$ is conjugate to $\langle a,a^jb\rangle$

□

Theorem 2.11. *Let n be an even positive integer number, the normalitiy degree elements table of the dihedral group D_{2n} is given by the following table:*

η	$\cdots$	a^j	$\cdots$	$\cdots$	a^jb	$\cdots$
$\vdots$						
a^i		$\eta(a^i,a^j)$			$\eta(a^i,a^jb)$	
$\vdots$						
a^ib		$\eta(a^ib,a^j)$			$\eta(a^ib,a^jb)$	
$\vdots$						

the entries is define by:

1. $\eta(a^i, a^j) = 1$ *for all* $1 \leq i, j \leq n$*;*

2. $\eta(a^i, a^j b) = \begin{cases} 1 & if\ gcd(i,n) = 1\ or\ 2 \\ 0 & Otheriwse \end{cases}$ *for all* $1 \leq i, j \leq n$*;*

3. $\eta(a^i b, a^j) = \eta(a^i, a^j b)$*;*

4. $\eta(a^i b, a^j b) = \begin{cases} 1 & if\ gcd(j-i,n) = 1 \\ 0 & Otheriwse \end{cases}$ *for all* $1 \leq i, j \leq n$.

Theorem 2.12. *Let* n *be a positive integer number, the normalitiy degree of the Dihedral group* D_{2n} *is given by the following:*

1. If n is an odd number, then $P_{rn}(D_{2n}) = \frac{n+3\varphi(n)}{4n}$;

2. If n is an even number, then $P_{rn}(D_{2n}) = \frac{n+3(\varphi(n)+\varphi(\frac{n}{2}))}{4n}$.

In this section we will presented the normal graph which is defined by the set of all vertices are elements group and for any two vertices x, y are adjacency if and only if $\langle x, y \rangle \trianglelefteq \mathcal{G}$ and we denoted by $\Gamma_G(N)$

Definition 2.13. Let G be a group. The normalizer of an element x of G, denoted $\mathbf{Nor}_G(x)$ defined by $\mathbf{Nor}_G(x) = \{y | \langle x, y \rangle \trianglelefteq G\}$.

Proposition 2.14. *Let* $n = 2^r \prod_{t=}^{s} p_t^{\alpha_t}$ *be an integer number, the set of neighbourhood of vertices of normal graph of group* D_{2n} *are given by the following:*

1) $\mathbf{Nor}_{D_{2n}}(a^i) = \begin{cases} \{y | \forall y \in D_{2n}\} & p_t \nmid i \quad (\varphi(n) + \varphi(\frac{n}{2})) \\ \{\{a^j\}\} & p_t \mid i \quad (n - (\varphi(n) + \varphi(\frac{n}{2}))) \end{cases}$;

2) $\mathbf{Nor}_{D_{2n}}(a^i b) = \begin{cases} \{a^j\} & 4p_s \nmid i, \forall s \\ \{a^{j+1} b\} & \end{cases}$.

Proposition 2.15. *Let* $n = 2^r \prod_{1=1}^{s} p_i^{\alpha_i}$ *be positive integer number. The sequence degree of vertices in normal graph of the group* D_{2n} *and* $D_{2n} \times C_p$ *are given by the following:*

1) If $2 \nmid n$, then:

$$d(\Gamma_{D_{2n}}(N)) = \begin{pmatrix} 2\varphi(n) & 2n-1 & n-1 \\ n & \varphi(n) & n-\varphi(n) \end{pmatrix}$$

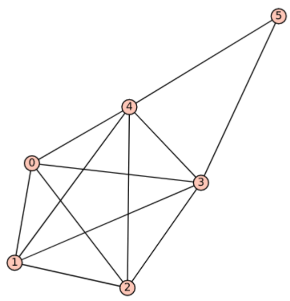
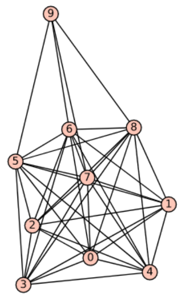
2) If $2 \mid n$, then:

$$d(\Gamma_{D_{2n}}(N)) = \begin{pmatrix} 2(\varphi(n) + \varphi(\frac{n}{2})) & 2n-1 & n-1 \\ n & \varphi(n) + \varphi(\frac{n}{2}) & n - (\varphi(n) + \varphi(\frac{n}{2})) \end{pmatrix}$$

Corollary 2.16. *The following are holds:*

$$|E(\Gamma_G(N))| = \begin{cases} \frac{n(3\varphi(n)-1)+n^2}{2} & G \cong D_{2n}, 2 \nmid n \\ \frac{n(n+3(\varphi(n)+\varphi(\frac{n}{2}))-1)}{2} & G \cong D_{2n}, 2 \mid n \end{cases}.$$

For example:

G	D_6	D_{10}
$\Gamma_G(N)$		
$d(\Gamma)$	$\begin{pmatrix}4 & 5 & 2\\ 3 & 2 & 1\end{pmatrix}$	$\begin{pmatrix}8 & 9 & 4\\ 5 & 4 & 1\end{pmatrix}$
$\lvert E(\Gamma_G(N))\rvert$	12	40

2.1 The relation between $P_{nor}(G)$ and $\Gamma_N(G)$.

We begin with the following Proposition.

Proposition 2.17. *Let Γ_N be a simple and normal graph, The number of degree vertices $deg(v)$ for any $v \in V(\Gamma_N)$ when $G \cong D_{2n}$ is given by the following:*

1. *if $v = a^i$, then $deg(v) = |N_G(v)| - 1$;*
2. *if $v = a^i b$, then $deg(v) = |N_G(v)|$;*

Proof. Let $v = a^i$ for $1 \leq i \leq n$ be a vertex, It is clear that the $N_G(v) = \{u \in G \mid \langle v, u\rangle \trianglelefteq G\}$ and $deg(v)$ represents the number of vertices from G which are adjacent to v. Since $v \in N_G(v)$, therefore $|N_G(v)| - 1$ represents the number of vertices which are adjacent to v. Thus $deg(v) = |N_G(u)| - 1$. If $v = a^i b$ it is clear that $a^i b \notin N_G(a^i b)$. □

Theorem 2.18.

$$P_{nor}(G) = \frac{2|E(\Gamma_N)| + n}{|G|^2}$$

Proof. In the first, the parameters normality degree is define by $P_{nor}(G) = \dfrac{\{(u, v) \in G \times G, \langle u, v\rangle \trianglelefteq G\}}{|G|^2}$, Let $|G| = n$, suppose that u_i and u_j are elements in G and $cl_G(u_i)$ where $1 \leq i \leq r$, we can used this definition by

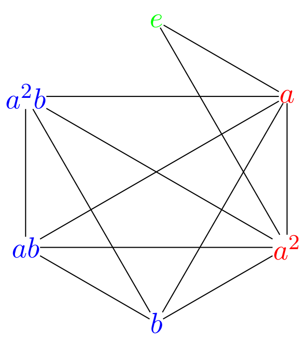
$$\begin{aligned}
P_{nor}(G) &= \frac{|\{(u_i, u_j) \in G \times G, \langle u_i, u_j\rangle \trianglelefteq G\}|}{|G|^2} \\
&= \frac{|N_G(u_1) \cup N_G(u_2) \cup \cdots \cup N_G(u_n)|}{|G|^2} \\
&= \frac{|N_G(u_1)| + |N_G(u_2)| + \cdots + |N_G(u_n)|}{|G|^2} \\
&= \frac{|cl_g(u_1)||N_G(u_1)| + |cl_g(u_2)||N_G(u_2)| + \cdots + |cl_g(u_r)||N_G(u_r)|}{|G|^2} \\
&= \frac{\sum_{1 \leq i \leq r} |cl_g(u_i)||N_G(u_i)|}{|G|^2} \\
&= \frac{2|E(\Gamma_N)|}{|G|^2}.
\end{aligned}$$

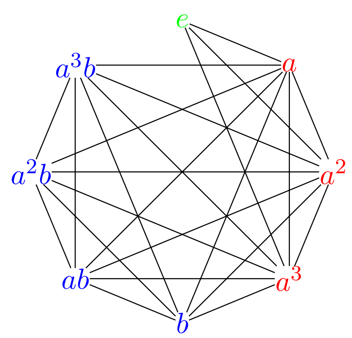
□

Example 2.19. In this example we will present two cases to compute the normalitiy degree table and normal graph.

D_6	e	a	a^2	b	ab	a^2b
e	1	1	1	0	0	0
a	1	1	1	1	1	1
a^2	1	1	1	1	1	1
b	0	1	1	0	1	1
ab	0	1	1	1	0	1
a^2b	0	1	1	1	1	0
$\|N_G(g)\|$	3	6	6	4	4	4
$\|deg(g)\|$	2	5	5	4	4	4

D_8	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	1	1	1	1	0	0	0	0
a	1	1	1	1	1	1	1	1
a^2	1	1	1	1	1	1	1	1
a^3	1	1	1	1	1	1	1	1
b	0	1	1	1	0	1	1	1
ab	0	1	1	1	1	0	1	1
a^2b	0	1	1	1	1	1	0	1
a^3b	0	1	1	1	1	1	1	0
$\|N_G(g)\|$	4	8	8	8	6	6	6	6
$\|deg(g)\|$	3	7	7	7	6	6	6	6





$|N_{D_6}(e)| = 3$, $|N_{D_6}(a)| = 6$, $|N_{D_6}(a^2)| = 6$, $|N_{D_6}(b)| = 4$, $|N_{D_6}(ab)| =$ and $|N_{D_6}(a^2b)| = 4$

$$P_{nr}(D_6) = \frac{\sum_{\forall g \in D_6} |N_{D_6}(g)|}{36} = \frac{27}{36} = \frac{3}{4}$$

$$P_{nr}(D_6) = \frac{2E(\Gamma_N(D_6)) + 3}{36} = \frac{24+3}{36} = \frac{3}{4}$$

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