14th Iranian International Group Theory Conference

Action of automorphisms on complex irreducible characters: groups of type A

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Abstract

In the study of representations of finite reductive groups, Jordan decomposition of characters associates to each character in a rational Lusztig series a unipotent character of the centralizer of a semi-simple element in the dual group. In this paper we report our recent work on how and in which sense one can construct a Jordan decomposition of characters of $\mathbf{G}^F$ which is equivariant with respect to $\mathrm{Aut}(\mathbf{G}^F)$ whenever $\mathbf{G}$ is a simple simply-connected algebraic group of type A defined over a finite field. This is part of the project to determine the action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}(G)$ for all finite (quasi-)simple groups G.

Keywords: Irreducible characters, Automorphisms
Mathematics Subject Classification [2010]: 20C15, 20C05

1 Introduction

A finite reductive group is the fixed-point subgroup $G := \mathbf{G}^F$ of a connected reductive algebraic group $\mathbf{G}$ defined over the finite field $\mathbb{F}_q$ of characteristic $p > 0$, where $F : \mathbf{G} \to \mathbf{G}$ is the Frobenius map corresponding to this $\mathbb{F}_q$-structure. In recent years, many conjectures in representation theory of finite reductive groups have been "reduced" to check some new technical conditions about quasi-simple groups of Lie type. These new conditions to check involve analysing the action of automorphisms of a quasi-simple Lie-type group on the set of its irreducible characters.

Question. ([5, Problem 2.33]) For G a quasi-simple group of Lie type, determine the action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}(G)$.

For $G = \mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$, we first determine the action of automorphisms on irreducible characters of G. Then, considering a larger framework, we determine the action of automorphisms on irreducible characters of G, for G being a general finite reductive group of type A. Using the usual Gelfand-Graev characters of G, Brunat and Himstedt turned the action of automorphisms on regular characters to the action on the corresponding labels (s, ξ), see [1, 2]. In this paper, using a regular embedding $\mathbf{G} \to \widetilde{\mathbf{G}}$ into a connected-center group $\widetilde{\mathbf{G}}$, it is shown that the various components of restrictions of irreducible characters of $\widetilde{\mathbf{G}}$ to $\mathbf{G}$ can be distinguished by the G-classes of their unipotent support which are equivarant under the action of automorphisms. This generalizes a recent result of [3] and provides a useful tool for investigating local-global conjectures as one usually need to deal with Levi subgroups.

[1]speaker

1.1 Notation.

We denote by $\mathrm{Res}^G_H\chi$ the restriction of a character χ of G to some subgroup $H \leq G$. Also the induction of a character ψ of H to G is denoted by $\mathrm{Ind}^G_H\psi$. For $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$, we denote by $\mathrm{Irr}(N|\chi)$ the set of irreducible constituents of the restriction $\mathrm{Res}^G_N\chi$. The stabilizer of $\psi \in \mathrm{Irr}(N)$ under the action of G on $\mathrm{Irr}(N)$ is denoted by G_ψ. We also write $\mathrm{Aut}(\mathbf{G}, F)$ for the set of automorphisms of $\mathbf{G}$ commuting with the Frobenius map F. Other notation are standard or will be defined where needed.

2 Main results

Let $\mathbf{G}$ be a connected reductive algebraic group defined over an algebraic closure $\mathbb{K} = \overline{\mathbb{F}_p}$ of the finite field of prime order p and let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism defining an $\mathbb{F}_q$-rational structure $G = \mathbf{G}^F$ on $\mathbf{G}$. Assuming p is a good prime for $\mathbf{G}$ a theory of generalised Gelfand–Graev characters (GGGCs) was developed by Kawanaka in [4]. These are certain characters Γ_u of G which are defined for any unipotent element $u \in G$. Note that $\Gamma_u = \Gamma_v$ whenever $u, v \in G$ are G-conjugate so the GGGCs are naturally indexed by the unipotent conjugacy classes of G.

It has been shown by Lusztig that the irreducible characters of $G = \mathbf{G}^F$ can be partitioned into the so-called geometric Lusztig series, labelled by the semisimple $\mathbf{G}^*$-classes of $\mathbf{G}^{*F^*}$, where $(\mathbf{G}^*, F^*)$ denotes a pair dual to $(\mathbf{G}, F)$. If such a series is labelled by a semisimple class with representative s, then it contains $|A_{\mathbf{G}^*}(s)^{F^*}|$ semisimple characters, where $A_{\mathbf{G}^*}(s) = C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$ is the component group of s. The set of semisimple (and regular) characters of G can be naturally parametrized by pairs (s, ξ) where s runs over a set of representatives of the semisimple classes of $G^* = \mathbf{G}^{*F^*}$ and $\xi \in \mathrm{Irr}(A_{G^*}(s))$, where $A_{G^*}(s) := A_{\mathbf{G}^*}(s)^{F^*}$. For any $\lambda \in \mathrm{Irr}(C^\circ_{G^*}(s))$, we denote by $A_{G^*}(s)_\lambda$ the stabilizer of λ under $A_{G^*}(s)$. The main goal of this paper is to show the following.

Theorem 2.1. *[6] Assume that $G = \mathrm{SL}^\epsilon_n(q)$. Then, for any semisimple element $s \in G^*$ and any unipotent character $\lambda \in \mathrm{Irr}(C^\circ_{G^*}(s))$, there exists a morphism $\omega^0_{s,\lambda} : H^1(F, Z(\mathbf{G})) \to \mathrm{Irr}(A_{G^*}(s)_\lambda)$ such that for the irreducible character $\chi_{s,\lambda,\omega^0_{s,\lambda}(z)} \in \mathrm{Irr}(G)$, parametrized by triple $(s, \lambda, \omega^0_{s,\lambda}(z))$ for some $z \in H^1(F, Z(\mathbf{G}))$, one has*

$$^{\sigma}\chi_{s,\lambda,\omega^0_{s,\lambda}(z)} = \chi_{\sigma^{*-1}(s),\sigma^*(\lambda),\omega^0_{s,\lambda}(\sigma(z))},$$

where $\sigma \in \langle F_p, \gamma\rangle \leq \mathrm{Out}(G)$ and $\sigma^* \in \mathrm{Aut}(G^*)$ is its dual automorphism.

Using the above theorem, we obtain a short proof of the stabilizer condition in the so-called inductive McKay condition for the irreducible characters of G.

Corollary 2.2. *If $\widetilde{\chi} = \chi_{\widetilde{s},\lambda} \in \mathrm{Irr}(\widetilde{G})$, then for the irreducible character $\chi_0 = \chi_{s,\lambda,1} \in \mathrm{Irr}(G|\widetilde{\chi})$ we have*

$$(\widetilde{G} \times \langle F_p, \gamma\rangle)_{\chi_0} = \widetilde{G}_{\chi_0} \times (\langle F_p, \gamma\rangle)_{\chi_0}.$$

Acknowledgment

This work was supported by the Grant No. 98012009 from the Iranian National Science Foundation (INSF).

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