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On Strongly Connected Partial Hypergraph

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Abstract

In this paper, we study the concept of strongly connected hypergraphs as a generalization of strongly connected graphs. We also get the connection between strongly connected hypergraph and the associated path hyperoperation and investigate their basic properties.

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1 Introduction

Hypergraphs are the generalization of graphs (see [2]) in case of set of multiary relations. It means the expansion of graph models for the modeling complex systems. Algebraic hyperstructures, in particular hypergroups, were introduced in 1934 by Marty, at the 8th Congress of Scandinavian Mathematicians (see [11]) and then it was developed by many researchers. Since then, hundreds of papers and several books have been written on this topic. Nowadays, there are many connections between hyperstructures and other branches of mathematics, leading to applications in hypergraphs, binary relations, combinatorics, artificial intelligence, automata and fuzzy sets. One can find a survey of hyperstructure theory and their applications in [5, 6]. The concept of hypergroupoids deriving from binary relations, namely *C-hypergroupoids* were delineated by Corsini in [3]. Also, the connection between hyperstructures and binary relations in general, has been investigated in [6, 7].

A new class of hyperoperations, namely path hyperoperations that are obtained from binary relations and their connections with graph theory, were introduced by Kalampakas et al. [8, 10].

In this paper, we study the concept of strongly connected hypergraphs and investigate the connection between strongly connected hypergraph and the associated path hyperoperation.

2 Preliminary Results

In this section, we introduce some preliminary results and definitions which will be needed in the subsequent sections. We provide some definitions from the theory of hypergraphs. The interested reader should refer to [2] for more concepts of hypergraph theory.

A *hypergraph* Γ is a pair (V, E), where where V is a finite set of vertices and E is a set of hyperedges which are arbitrary nonempty subsets of V such that $\bigcup_j E_j = V$. A hypergraph is a generalization of an ordinary undirected graph, such that an edge need not contain exactly two nodes, but can instead contain an arbitrary nonzero number of vertices. Two vertices u and v are *adjacent* in Γ=(V, E) if there is an edge

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$e \in E$ such that $u, v \in e$. If for two edges $e, f \in E$, $e \cap f \neq \emptyset$, then we say that e and f are *adjacent*. A vertex v and an edge e are *incident* if $v \in e$. We denote by $\Gamma(v)$ the *neighborhood* of a vertex v, i.e. $\Gamma(v) = \{u \in V : \{u, v\} \in E\}$. Given $v \in V$, denote the number of edges incident with v by $d(v)$; $d(v)$ is called the *degree* of v. A hypergraph in which all vertices have the same degree d is said to be *regular* of degree d or *d-regular*. The size, or the *cardinality*, $|e|$ of a hyperedge is the number of vertices in e. A hypergraph Γ is *simple* if there are no repeated edges and no edge properly contains another. A hypergraph is known as *uniform* or *k-uniform* if all the edges have cardinality k. Note that an ordinary graph with no isolated vertices is a 2-uniform hypergraph.

A *partial hypergraph* $\Gamma' = (V', E')$ of a hypergraph $\Gamma = (V, E)$, denoted by $\Gamma' \subseteq \Gamma$, is a hypergraph such that $V' \subseteq V$ and $E' \subseteq E$. The partial hypergraph $\Gamma' = (V', E')$ is *induced* if $E' = \{e \in E \mid e \subseteq V'\}$. Induced hypergraphs will be denoted by $\langle V' \rangle$. A partial hypergraph of a simple hypergraph is always simple.

Let $\Gamma = (V, E)$ be a hypergraph. A *path* of length k in Γ is an alternating sequence $P_{v_1, v_{k+1}} = (v_1, e_1, v_2, ..., v_k, e_k, v_{k+1})$ in which $v_i \in V$ for each $i = 1, 2, ..., k+1$, $e_i \in E$, $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 1, 2, ..., k$ and $v_i \neq v_j$, $e_i \neq e_j$ for $i \neq j$. Also, a hypergraph is *connected* if there is a path between every pair of vertices. A *connected component* of a hypergraph is every maximal set of vertices such that are pairwise connected by a path. A *cycle* of length k is a sequence $(v_1, e_1, v_2, ..., v_k, e_k, v_1)$, such that P_{v_1, v_k} is a path. Also, a hypergraph is called *acyclic* if it does not contain any cycles.

Definition 2.1. Let H be a non-void set and $P^*(H)$ be the set of all non-void subsets of H. A *hyperoperation* on H is a map $* : H^2 \to P^*(H)$ and the couple $(H, *)$ is called a *partial hypergroupoid.* The structure $(H, *)$ is called a *non-partial hypergroupoid* if for every $x, y \in H$ we have $x * y \neq \emptyset$. A hypergroupoid $(H, *)$ is called a *total hypergroupoid* if $\forall x, y \in H,\ x * y = H$. A hypergroupoid $(H, *)$ is called *commutative* if for all $x, y \in H$ we have $x * y = y * x$. Also, $(H, *)$ is called *weakly commutative* if $\forall x, y \in H, \quad x * y \cap y * x \neq \emptyset$.

Definition 2.2. If A and B are non-void subsets of H, then $A * B$ is defined by,

$$A * B = \bigcup_{a \in A, b \in B} a * b.$$

(i) A *semihypergroup* is a hypergroupoid $(H, *)$ which satisfies the associative axiom:

$$\forall x, y, z \in H, \quad (x * y) * z = x * (y * z).$$

(ii) A *quasihypergroup* is a hypergroupoid $(H, *)$ which satisfies the reproductive axiom:

$$\forall x \in H, \quad x * H = H = H * x.$$

(iii) A *hypergroup* is a semihypergroup which is also a quasihypergroup.

A non-void subset K of a hypergroup $(H, *)$ is called a *subhypergroup* if it satisfies the reproductive axiom, i.e., for all $k \in K$, $k * K = K * k = K$.

Let $(H_1, *_1)$ and $(H_2, *_2)$ be two hypergroupoids. A map $f : H_1 \to H_2$ is called a *homomorphism* if $\forall x, y \in H_1,\ f(x *_1 y) \subseteq f(x) *_2 f(y)$ and it is called a *good homomorphism* if for all $x, y \in H_1,\ f(x *_1 y) = f(x) *_2 f(y)$.

We say that the two hypergroups H_1 and H_2 are *isomorphic* if there is a good homomorphism between them which is also a bijection and we write $H_1 \cong H_2$.

The relationship between hyperstructure theory and hypergraph theory has been studied by many authors (see [1, 4]).

Definition 2.3. Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph. The hypergroupoid $H_\Gamma = (H, \circ)$ such that the hyperoperation " $\circ$ " on H is defined as follows:

$$\forall (x, y) \in H^2, \quad x \circ y = E(x) \cup E(y),$$

is called a *hypergraph hypergroupoid*, where $E(x) = \bigcup_{x \in E_i} E_i$.

Let $E \subseteq H \times H$ be a binary relation. The Corsini's hyperoperation $\bullet_E : H \times H \to P^*(H)$ is defined as follows:

$$(x, y) \mapsto x \bullet_E y := \{z \in H \mid (x, z) \in E \text{ and } (z, y) \in E\}.$$

The hypergroupoid $(H, \bullet_E)$ is called *Corsini's partial hypergroupoid* or simply *partial C-hypergroupoid* associated with the binary relation on H (see [3, 9]). If $x \bullet_E y \neq \emptyset$ for all $x, y \in H$, then $(H, \bullet_E)$ is called C-hypergroupoid. Clearly, a partial C-hypergroupoid $(H, \bullet_E)$ is a C-hypergroupoid if and only if $E \circ E = H^2$, so that "$\circ$" is the usual relation composition.

Definition 2.4. Let $\Gamma = (V, E)$ be a hypergraph. We define the *path hyperoperation* $\circ_\Gamma : V \times V \to P^*(V)$ for all $x, y \in V$ as follows:

$$x \circ_\Gamma y := \{z \in V \mid z \text{ belongs to a path from } x \text{ to } y\}.$$

The (partial) hypergroupoid $(V, \circ_\Gamma)$ is called the (partial) *path hypergroupoid* corresponding with Γ.

The hyperoperation "$\circ_\Gamma$" on V is called a *non-partial hyperoperation* if for all $x, y \in V$ we have $x \circ_\Gamma y \neq \emptyset$. In this case, the path hypergroupoid associated with Γ is called non-partial.

It is easy to check that, for any hypergraph $\Gamma = (V, E)$ and for all $x, y \in V$, the Corsini product $x \bullet_E y$ is a subset of $x \circ_\Gamma y$ i.e., $x \bullet_E y \subseteq x \circ_\Gamma y$.

3 Main results

In this section, we study the concept of strongly connected hypergraphs as a generalization of strongly connected graphs. We also investigate the connection between strongly connected hypergraph and the associated path hyperoperation. By the definition of the path in a hypergraph we obtain the following results.

Proposition 3.1. *Let $\Gamma = (V, E)$ be a hypergraph. Then for any $x, y \in V$, $x \circ_\Gamma y \neq \emptyset$ if and only if there exists a path from x to y.*

Proposition 3.2. *([9]) Let $G = (V, E)$ be a graph, then the Corsini hyperoperation "$\bullet_E$" associated with G, is non-partial if and only if there exists a path with length 2 between any pair of vertices of G.*

This result allows us to prove the following corollary.

Corollary 3.3. *Let $\Gamma = (V, E)$ be a hypergraph and let "$\circ_\Gamma$" be the associated path hyperoperation with Γ. Then "$\circ_\Gamma$" is a non-partial hyperoperation if and only if for any $x, y \in V$, there exists a path from x to y.*

Proof. The path hyperoperation associated with Γ is a partial hyperoperation if and only if $x \circ_\Gamma y \neq \emptyset$ for all $x, y \in V$. Therefore, by Proposition 3.1, $x \circ_\Gamma y \neq \emptyset$ if and only if there exists a path from x to y. □

Directed hypergraphs is a generalization of directed graphs (digraphs). Directed hypergraphs modelling can be very useful in formal language theory, relational database theory, scheduling and many other fields. A directed hypergraph is defined as follows.

Definition 3.4. A *directed hypergraph* Γ (or *dihypergraph*) is a pair (V, E), where V is a finite set of vertices and E is a set of *hyperarcs*. A hyperarc E is an ordered pair (T, H) of disjoint subsets of V. The set T is the *tail set* of the hyperarc, while H is called the *head set* of the hyperarc.

The *size* of a dihypergraph Γ is defined as $|\Gamma| = \Sigma_{e \in E} |tail(e)|$. An example of a directed hypergraph is illustrated in Figure 1.

In the following definition we state the notion of strong connectivity of hypergraphs. This concept is a generalization of strong connectivity of graphs. Hypergraph connectivity can be used in networking, mobile communication systems, shortest path, database theory, image processing and numerous other applications.

Definition 3.5. A dihypergraph $\Gamma = (V, E)$ is called *strongly connected* if for every $x, y \in V$ there exists at least one directed path between x and y.

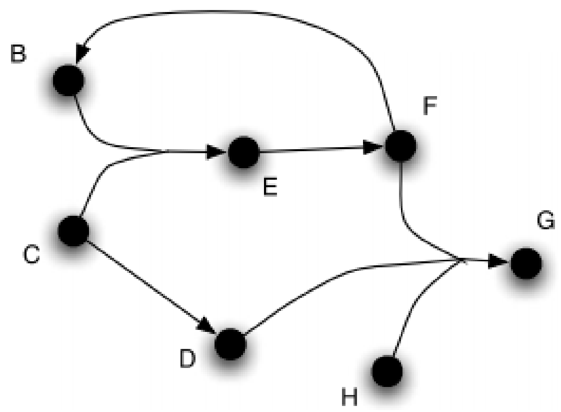

Figure 1: A directed hypergraph

In the sequel we study the connection between the path hyperoperation and hypergraph connectivity.

Theorem 3.6. *Let $\Gamma = (V, E)$ be a dihypergraph. Then Γ is strongly connected if and only if the associated path hyperoperation is non-partial.*

Proof. Suppose that Γ is a strongly connected hypergraph. Then for any $x, y \in V$ there exists a path between x and y. By Proposition 3.1 we have $x \circ_\Gamma y \neq \emptyset$ for every $x, y \in V$. Thus the associated path hyperoperation with Γ is non-partial.

Conversely, suppose that the associated path hyperoperation with Γ, "$\circ_\Gamma$" be non-partial. So, for any $x, y \in V$ we have $x \circ_\Gamma y \neq \emptyset$. Therefore, by Proposition 3.1 there exists a path between x and y. Thus Γ is strongly connected. □

Definition 3.7. For a given dihypergraph $\Gamma = (V, E)$, the strongly connected component of Γ is called the *strongly connected partial hypergraph*. More exactly, a strongly connected component of a hypergraph $\Gamma = (V, E)$ is a maximal partial hypergraph $\Gamma' = (V', E')$ such that there exists a path between every two vertices of V'.

Thus we have the following result.

Proposition 3.8. *Let $\Gamma = (V, E)$ be a dihypergraph and $x, y \in V$. Then $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma x \neq \emptyset$ if and only if x and y belong to the common strongly connected component of Γ.*

Proof. By Theorem 3.6, the proof is obvious. □

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