14th Iranian International Group Theory Conference

The vanishing conjugacy class sizes and solvability of a finite group

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Abstract

Let G be a finite group and $\emptyset \neq \pi \subseteq \pi(G)$. An element $g \in G$ is called a vanishing element of G if there exists an irreducible complex character χ of G such that $\chi(g) = 0$. In this case, the size of the conjugacy class of G containing g is called a vanishing conjugacy class size of G. In this talk, we concern on the structure of the finite groups with exactly one vanishing conjugacy class size and the structure of the finite groups with exactly one conjugacy class size of π-elements.

Keywords: Vanishing elements, the irreducible characters, the conjugacy class sizes
Mathematics Subject Classification [2010]: 20E45;20C15;20D05

1 Introduction

Throughout this paper, G is a finite group, $Z(G)$ is the center of G and $\mathrm{Fit}(G)$ is the Fitting subgroup of G. For $a \in G$, $cl_G(a)$ is the conjugacy class in G containing a and $C_G(a)$ denotes the centralizer of the element a in G.

An element $g \in G$ is called as vanishing element of G if there exists an irreducible complex character χ of G such that $\chi(g) = 0$, otherwise, g is non-vanishing in G. A conjugacy class $cl_G(a)$ of G is vanishing if $cl_G(a)$ contains a vanishing element and if $cl_G(a)$ is a vanishing conjugacy class, then the size of $cl_G(a)$, denoted by $|cl_G(a)|$, is called a vanishing conjugacy class size. We denote by $\mathrm{Van}(G)$ the set of the vanishing elements of G. Let $\pi(G)$ be the set of prime divisors of $|G|$. For instance, $\mathrm{Van}(\mathrm{Alt}_4) = cl_{\mathrm{Alt}_4}((1\ 2\ 3)) \cup cl_{\mathrm{Alt}_4}((1\ 3\ 2))$. Hence, Alt_5 has exactly one vanishing conjugacy class size which is 4. So, 3 does not divide any vanishing conjugacy class sizes of Alt_4. In [7], the authors showed that if a prime p does not divide any vanishing conjugacy class sizes of G, then G has a normal p-complement, as you can see in Alt_4. Also, looking at the character table of Alt_4 may motivate us to consider the finite groups with exactly one vanishing conjugacy class size. In [5], Bianchi, Camina, Lewis and Pacifici classify the finite super-solvable groups with one vanishing conjugacy class size. They put forward a problem on the solvability of the finite groups with one vanishing conjugacy class size. In [1], we give an affirmative answer to this problem. In a part of this talk, we will concern on this interest.

On the other hand, studying the existence of Hall subgroups (in particular, by considering the character table) is a classical subject in the finite group theory and it has been considered in many articles, for instance, see [6, Problem 11] and [3]. Moreover, by Wielandt's theorem [11, 9.1.10], if G has a nilpotent Hall π-subgroup for a set π of primes, then the set of Hall π-subgroups of G is a conjugacy class and every π-subgroup is contained in some Hall π-subgroup. This makes studying whether G has a nilpotent Hall π-subgroup is an interesting subject in the finite group theory.

[1]speaker

In 2015, Beltrán, Felipe and Shao in [4] showed that if p is a prime and N is a non-trivial normal subgroup of G such that for every p'-element $x \in N - Z(N)$, $|cl_G(x)| = m$, then N has nilpotent p-complements. In [2], we show that if for every π-element $x \in N - Z(N)$, $|cl_G(x)| = m$, then N has nilpotent Hall π-subgroups. This result is a key one in studying the finite groups with exactly one vanishing conjugacy class size in [1]. So, as another part of this talk, we will concern on this result.

2 Main Results

In [10, Theorem], M. Isaacs proved that:

Proposition 2.1. [10, Theorem] *Let N be a normal subgroup of G. Suppose that all of the conjugacy classes of G which lie outside N have equal sizes. Then, either G/N is cyclic or else every non-identity element of G/N has prime order.*

Let N be a normal subgroup of a finite group G. By the above proposition, $G/N \cong \mathrm{Alt}_5$ can be a possibility. Also, we do not know whether all of the elements of G which lie outside $\mathrm{Fit}(G)$ are vanishing in G. So, we can not apply Proposition 2.1 to study the finite groups with exactly one vanishing conjugacy class size. However, we know that:

Proposition 2.2. *Let $g \in G$.*

(i) [9] *If G is solvable and $g \notin \mathrm{Fit}(G)$ is non-vanishing in G, then* $\gcd(2, o(g\mathrm{Fit}(G))) \neq 1$.

(ii) [8] *If $g \notin \mathrm{Fit}(G)$ is non-vanishing in G, then* $\gcd(6, o(g\mathrm{Fit}(G))) \neq 1$.

Also, in [1, Proposition 3], we prove that:

Theorem 2.3. [1, Proposition 3] *Let $N \neq \{1\}$ be a normal solvable subgroup of G and, let $x \in N$ be non-vanishing in G. If $o(x\mathrm{Fit}(G))$ is odd in $G/\mathrm{Fit}(G)$, then $x \in \mathrm{Fit}(G)$.*

The other key facts needed in study of finite groups with exactly one vanishing conjugacy class size are the following ones:

Theorem 2.4. [2, Theorem] *Let N be a non-trivial normal subgroup of G and $\emptyset \neq \pi \subseteq \pi(N)$. If $2 \notin \pi$ and for every π-element $x \in N - Z(N)$ with $|\pi(x)| \leq 2$, $|cl_G(x)| = m$, for some integer m, then N has nilpotent Hall π-subgroups. In particular, either $|\pi \cap \pi(N/Z(N))| \leq 1$ or N has abelian Hall π-subgroups.*

There are some examples which show that the assumption $2 \notin \pi$ in Theorem 2.4 is essential. However, if we restrict our attention to the solvable groups, then the assumption $2 \notin \pi$ in Theorem 2.4 can be removed and we can prove that:

Corollary 2.5. *Let N be a non-trivial normal subgroup of a solvable group G and $\emptyset \neq \pi \subseteq \pi(N)$. If for every π-element $x \in N - Z(N)$ with $|\pi(x)| \leq 2$, $|cl_G(x)| = m$, for some integer m, then N has nilpotent Hall π-subgroups.*

Now, applying Proposition 2.2 and Theorems 2.3 and 2.4 leads us to prove that:

Theorem 2.6. [1, Theorem A] *If G is a finite group with exactly one vanishing conjugacy class size, then G is solvable.*

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