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Influence of the fundamental group on incomplete lifting and its application

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Abstract

In this talk, after reviewing the concepts of continuous lifting of paths (homotopies), covering maps and fundamental groups, first we mention a result on incomplete lifting for local homeomorphism. Second, we prove some of the well-known properties of covering maps for local homeomorphisms. Also, we investigate the influence of the fundamental group on incomplete lifting.

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1 Introduction

J. Brazas [1, Definition 3.1] generalized the concept of covering map by the phrase *"A semicovering map is a local homeomorphism with continuous lifting of paths and homotopies"*. Note that a map $p : Y \to X$ has **continuous lifting of paths** if $\rho_p : (\rho Y)_y \to (\rho X)_{p(y)}$ defined by $\rho_p(\alpha) = p \circ \alpha$ is a homeomorphism for all $y \in Y$, where $(\rho Y)_y = \{\alpha : I = [0,1] \to Y | \alpha(0) = y\}$. Also A map $p : Y \to X$ **has continuous lifting of homotopies** if $\Phi_p : (\Phi Y)_y \to (\Phi X)_{p(y)}$ defined by $\Phi_p(\phi) = p \circ \phi$ is a homeomorphism for all $y \in Y$, where elements of $(\Phi Y)_y$ are endpoint preserving homotopies of paths starting at y. (see [2])

In this paper, all maps $f : X \to Y$ between topological spaces X and Y are continuous functions. We recall that a continuous map $p : \widetilde{X} \to X$ is called a *local homeomorphism* if for every point $\tilde{x} \in \widetilde{X}$ there exists an open neighborhood $\tilde{W}$ of $\tilde{x}$ such that $p(\tilde{W}) \subset X$ is open and the restriction map $p|_{\tilde{W}} : \tilde{W} \to p(\tilde{W})$ is a homeomorphism. In this paper, we denote a local homeomorphism $p : \widetilde{X} \to X$ by $(\widetilde{X}, p)$ and assume that $\widetilde{X}$ is path connected and p is surjective.

Definition 1.1. ([4]). Let $\widetilde{X}$ and X be topological spaces and let $p : \widetilde{X} \to X$ be continuous. An open set U in X is **evenly covered** by p if $p^{-1}(U)$ is a disjoint union of open sets S_i in $\widetilde{X}$, called **sheets**, with $p|_{S_i} : S_i \to U$ a homeomorphism for every i.

Definition 1.2. ([4]). If X is a topological space, then an ordered pair $(\widetilde{X}, p)$ is a **covering space** of X if:

1. $\widetilde{X}$ is a path connected topological space;
2. $p : \widetilde{X} \to X$ is continuous;

[1] speaker

3. each $x \in X$ has an open neighborhood $U = U_x$ that is evenly covered by p.

For a topological space X, by a path in X we mean a continuous map $\alpha : [0;1] \to X$. The points $\alpha(0)$ and $\alpha(1)$ are called the initial point and the terminal point of α, respectively. A loop α is a path with $\alpha(0) = \alpha(1)$. For a path $\alpha : [0;1] \to X$, α^{-1} denotes a path such that $\alpha^{-1}(t) = \alpha(1-t)$, for all $t \in [0,1]$. Denote $[0,1]$ by I, two paths $\alpha, \beta : I \to X$ with the same initial and terminal points are called *homotopic relative to end points* if there exists a continuous map $F : I \times I \to X$ such that

$$F(t,s) = \begin{cases} \alpha(t) & s = 0 \\ \beta(t) & s = 1 \\ \alpha(0) = \beta(0) & t = 0 \\ \alpha(1) = \beta(1) & t = 1. \end{cases}$$

Homotopy relative to end points is an equivalent relation and the homotopy class containing a path α is denoted by $[\alpha]$. For paths $\alpha, \beta : I \to X$ with $\alpha(1) = \beta(0), \alpha * \beta$ denotes the concatenation of α and β that is a path from I to X such that

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

The set of all homotopy classes of loops relative to the end point x in X under the binary operation $[\alpha][\beta] = [\alpha * \beta]$ forms a group and called the fundamental group of X denoted by $\pi_1(X,x)$ (see[4]). The set of all loops with initial point x in X called the loop space of X denoted by $\Omega(X,x)$ (see [4]).

2 Main results

In this section, we obtained some conditions under which a local homeomorphism is a semicovering map. First, we intend to show that if $p : \widetilde{X} \to X$ is a local homeomorphism, $\widetilde{X}$ is Hausdorff and sequential compact, then p is a semicovering map. In order to do this, we are going to study a local homeomorphism with a path which has no lifting.

Lemma 2.1. *Let $p : \widetilde{X} \to X$ be a local homeomorphism, f be an arbitrary path in X and $\tilde{x_0} \in p^{-1}(f(0))$ such that there is no lifting of f starting at $\tilde{x_0}$. If $A_f = \{t \in I \mid f|_{[0,t]}$ has a lifting $\hat{f}_t$ on $[0,t]$ with $\hat{f}_t(0) = \tilde{x_0}\}$, then A_f is open and connected. Moreover, there exists $\alpha \in I$ such that $A_f = [0, \alpha)$.*

Proof. Let β be an arbitrary element of A_f. Since p is a local homeomorphism, there exists an open neighborhood W at $\hat{f}_\beta(\beta)$ such that $p|_W : W \to p(W)$ is a homeomorphism. Since $\hat{f}_\beta(\beta) \in W$, there exists an $\epsilon \in I$ such that $f[\beta, \beta+\epsilon]$ is a subset of $p(W)$. We can define a map $\hat{f}_{\beta+\epsilon}$ as follows:

$$\hat{f}_{\beta+\epsilon}(t) = \begin{cases} \hat{f}_\beta(t) & t \in [0,\beta] \\ p|_W^{-1}(f(t)) & t \in [\beta, \beta+\epsilon] \end{cases}.$$

Hence $(0, \beta+\epsilon)$ is a subset of A_f and so A_f is open.

Suppose $t, s \in A$. Without loss of generality we can suppose that $t \geq s$. By the definition of A_f, there exists $\hat{f}_t$ and so $[0,t]$ is a subset of A_f. Also every point between s and t belongs to A_f hence A_f is connected. Since A_f is open connected and $0 \in A_f$, there exists $\alpha \in I$ such that $A_f = [0, \alpha)$. □

Now, we prove the existence and uniqueness of a concept of a defective lifting.

Lemma 2.2. *let $p : \widetilde{X} \to X$ be a local homeomorphism with unique path lifting property, f be an arbitrary path in X and $\tilde{x_0} \in p^{-1}(f(0))$, such that there is no lifting of f starting at $\tilde{x}$. Then, using notation of the previous lemma, there exists a unique continuous map $\tilde{f}_\alpha : A_f = [0, \alpha) \to \widetilde{X}$ such that $po\tilde{f}_\alpha = f|_{[0,\alpha)}$.*

Proof. First, we defined $\tilde{f}_\alpha : A_f = [0,\alpha) \to \widetilde{X}$ by $\tilde{f}_\alpha(s) = \hat{f}_s(s)$. The map $\tilde{f}_\alpha$ is well define since if $s_1 = s_2$, then by unique path lifting property of p we have $\hat{f}_{s_1} = \hat{f}_{s_2}$ and so $\hat{f}_{s_1}(s_1) = \hat{f}_{s_2}(s_2)$ hence $\tilde{f}_\alpha(s_1) = \tilde{f}_\alpha(s_2)$. The map $\tilde{f}_\alpha$ is continuous since for any element s of A_f, $\hat{f}_{\frac{\alpha+s}{2}}$ is continuous at s and $\hat{f}_{\frac{\alpha+s}{2}} = \hat{f}_s$ on $[0,s]$. Thus there exists $\epsilon > 0$ such that $\tilde{f}_\alpha|_{(s-\epsilon,s+\epsilon)} = \hat{f}_{\frac{\alpha+s}{2}}|_{(s-\epsilon,s+\epsilon)}$. Hence $\tilde{f}_\alpha$ is continuous at s. For uniqueness, if there exists $\hat{f}_\alpha : [0,\alpha) \to \widetilde{X}$ such that $p \circ \hat{f}_\alpha = f|_{[0,\alpha)}$, then by unique path lifting property of $\widetilde{X}$ we must have $\tilde{f}_\alpha = \hat{f}_\alpha$. □

Definition 2.3. By Lemmas 2.1 and 2.2, we called $\tilde{f}_\alpha$ the *incomplete lifting* of f by p starting at $\tilde{x_0}$.

Note that every compact metric space is sequential compact. In the following, we present two semicovering maps on compact metric spaces.

Example 2.4. We show that $p : S^1 \times S^1 \longrightarrow S^1 \times S^1$ defined by $(x,y) \longrightarrow (x^n y^m, x^s y^t)$ is a semicovering map, where $m,n,s,t \in \mathbb{N}$ such that $\frac{n}{s} \neq \frac{m}{t}$. Let $\exp(\theta) = e^{2\pi i\theta}$, then we can consider p as $p(\exp(\alpha),\exp(\beta)) = (\exp(n\alpha + m\beta), \exp(s\alpha + t\beta))$. As a notation put $\exp(\gamma,\eta) = \{\exp(\theta) \in S^1 | \gamma \leq \theta \leq \eta\}$. Suppose $l = Max\{n,m,s,t\}$ and $U = (\exp(\alpha - \frac{\pi}{2l}, \alpha + \frac{\pi}{2l})) \times (\exp(\beta - \frac{\pi}{2l}, \beta + \frac{\pi}{2l}))$ is an open neighborhood of an element $(\exp(\alpha),\exp(\beta)) \in S^1 \times S^1$. It is clear that $p|_U : U \longrightarrow \exp(n(\alpha - \frac{\pi}{2l}) + m(\beta - \frac{\pi}{2l}), n(\alpha + \frac{\pi}{2l}) + m(\beta + \frac{\pi}{2l})) \times \exp(s(\alpha - \frac{\pi}{2l}) + t(\beta - \frac{\pi}{2l}), s(\alpha + \frac{\pi}{2l}) + t(\beta + \frac{\pi}{2l}))$ is a homeomorphism. Note that

$$(m(\alpha + \frac{\pi}{2l}) + n(\alpha + \frac{\pi}{2l})) - (m(\alpha - \frac{\pi}{2l}) + n(\alpha - \frac{\pi}{2l})) < \frac{m}{l}(\frac{\pi}{2} + \frac{\pi}{2}) + \frac{n}{l}(\frac{\pi}{2} + \frac{\pi}{2}) < 2\pi$$

and

$$(s(\beta + \frac{\pi}{2l}) + t(\beta + \frac{\pi}{2l})) - (s(\beta - \frac{\pi}{2l}) + t(\beta - \frac{\pi}{2l})) < \frac{s}{l}(\frac{\pi}{2} + \frac{\pi}{2}) + \frac{t}{l}(\frac{\pi}{2} + \frac{\pi}{2}) < 2\pi.$$

Therefore, if $p(\exp(\alpha_1),\exp(\beta_1)) = p((\exp(\alpha_2),\exp(\beta_2)))$, then
$\begin{cases} n\alpha_1 + m\beta_1 = n\alpha_2 + m\beta_2 \\ s\alpha_1 + t\beta_1 = s\alpha_2 + t\beta_2 \end{cases}$ so $\begin{cases} n(\alpha_1 - \alpha_2) = m(\beta_2 - \beta_1) \\ s(\alpha_1 - \alpha_2) = t(\beta_2 - \beta_1) \end{cases}$. Since $\frac{n}{s} \neq \frac{m}{t}$, we have $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Thus p is a local homeomorphism. Note that $S^1 \times S^1$ is a compact metric space and so it is sequential compact hence by Theorem 2.5 p is a semicovering map. It should be mentioned that every semicovering of a path-connected, locally path-connected, semilocally simply connected space is a covering. Hence p is a covering map. Note that finding an evenly covered neighborhood by p for an arbitrary element of $S^1 \times S^1$ does not seem to be an easy computational task.

Theorem 2.5. *If $\widetilde{X}$ is Hausdorff and sequential compact and $p : \widetilde{X} \longrightarrow X$ is a local homeomorphism, then p is a semicovering map.*

Example 2.6. In Figure 1, the map p transfer every ci,j to ci directly for $i \in \mathbb{N}$ and $1 \leq j \leq 4$. Since the domain of p is compact metric, it is a sequential compact space and using Theorem 2.5 we can conclude that p is a semicovering map.

Clearly the composition of two local homeomorphisms is a local homeomorphism hence by Theorem 2.5 we have the following corollary.

Corollary 2.7. *If $p_i : \widetilde{X}_i \longrightarrow \widetilde{X}_{i-1}$ for $i = 1,2$ are local homeomorphisms and $\widetilde{X}_2$ is Hausdorff and sequential compact, then $p_1 \circ p_2$ is a semicovering map.*

Chen and Wang [3, Theorem 1] showed that a closed local homeomorphism p from a Hausdorff space $\widetilde{X}$ onto a connected space X is a covering map, when there exists at least one point $x_0 \in X$ such that $|p^{-1}(x_0)| = k$, for some finite number k. In the following theorem, we extend this result for semicovering map without finiteness condition on any fiber.

Theorem 2.8. *Let p be a closed local homeomorphism from a Hausdorff space $\widetilde{X}$ onto a space X. Then p is a semicovering map.*

Remark 2.9. Note that the local homeomorphism $p : S^1 \times S^1 \longrightarrow S^1 \times S^1$, introduced in Example 2.4, is a closed map since $S^1 \times S^1$ is Hausdorff and compact. Thus using Theorem 2.8 we can obtain another proof to show that p is semicovering.

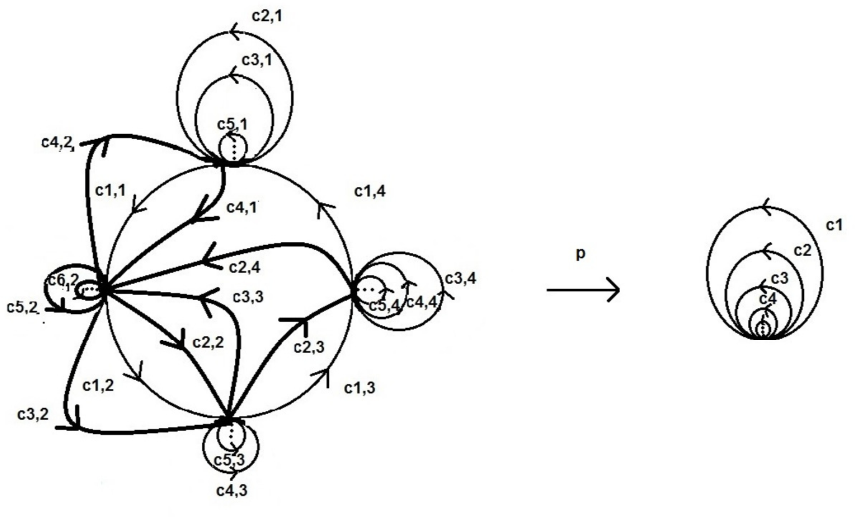

Figure 1: A semicovering map of a sequential compact space

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