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On the regularity of a divisibility graph for F-groups

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Abstract

The graph $D(G)$ is called the divisibility graph of G if its vertex set is the set of noncentral conjugacy class sizes of G and there is an edge between vertices a and b if and only if $a|b$ or $b|a$. We prove that, if the divisibility graph $D(G)$ where G is an F-group is a k-regular graph, then the divisibility graph $D(G)$ is a complete graph with $k+1$ vertices.

Keywords: conjugacy class, divisibility graph, F-group.

Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

There are some graphs related to finite groups and these graphs have been widely studied. In [4] A. R. Camina and R. D. Camina introduced a new graph. This graph is called divisibility graph $\overrightarrow{D}(X)$ for a set of positive integers X. Its vertex set is $V(\overrightarrow{D}(X)) = X^*$ and the edge set is $E(\overrightarrow{D}(X)) = \{(x, y); x, y \in X^*, x|y\}$. This graph is not strongly connected, so we consider simple graph $D(X)$ instead of $\overrightarrow{D}(X)$.Throughout the paper, G denotes a finite nonabelian group and x denotes an element of G. x^G denotes the G-conjugacy class containing x, $|x^G|$ denotes the size of x^G and $cs(G) = \{|x^G|; x \in G\}$ denotes the set of G-conjugacy class sizes and $cs^*(G) = cs(G) \setminus \{1\}$. Also we denote $D(G)$ instead of $D(cs(G))$. We denote by K_n the complete graph on n vertices, by $K_m + K_n$ a graph with connected components K_n and K_m.

In this paper, we prove that, if the divisibility graph $D(G)$ where G is an F-group is a k-regular graph, then the divisibility graph $D(G)$ is a complete graph with $k+1$ vertices. A finite group G is called an F-group, if for every $x, y \in G \setminus Z(G)$, $C_G(x) \leq C_G(y)$ implies that $C_G(x) = C_G(y)$.

2 Main results

In [6], the structure of nonabelian F-groups is given by J. Rebmann that we show this complete list below:

Theorem 2.1. *[6] Let G be a nonabelian group. Then G is an F-group if and only if it is one of the following types:*

(1) *G has an abelian normal subgroup of prime index;*

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(2) *$G/Z(G)$ is a Frobenius group with Frobenius kernel $L/Z(G)$ and Frobenius complement $K/Z(G)$, where L and K are abelian;*

(3) *$G/Z(G)$ is a Frobenius group with Frobenius kernel $L/Z(G)$ and Frobenius complement $K/Z(G)$ with K abelian, $Z(L) = Z(G)$, $L/Z(G)$ has prime power order and L is an F-group;*

(4) *$G/Z(G) \cong S_4$ and if $V/Z(G)$ is the Klein four-group in $G/Z(G)$, then V is nonabelian;*

(5) *$G \cong A \times P$ where P is an F-group of prime power order and A is abelian;*

(6) *$G/Z(G) \cong PSL(2,p^n)$ or $PGL(2,p^n)$, $G' \cong SL(2,p^n)$, where p is a prime and $p^n > 3$;*

(7) *$G/Z(G) \cong PSL(2,9)$ or $PGL(2,9)$ and G' is isomorphic to the Schur cover of $PSL(2,9)$.*

Theorem 2.2. *[5] Let G be a nonabelian F-group. Then the divisibility graph $D(G)$ is either disconnected or a star.*

Recalling that, given a positive integer k, a graph is called k-regular if every vertex is adjacent to exactly k vertices.

In [3], the authors posed a conjecture that states for a positive integer k, the graph $\Gamma(G)$ is k-regular if and only if it is a complete graph with $k+1$ vertices. They proved this conjecture for the case $k \leqslant 3$. Then in [2], the authors proved the conjecture for an F-group or $k \leqslant 5$. Eventually, the authors in [1] provided an affirmative answer to the conjecture in full generality.

Now, this question may come up to one person, is this conjecture true for the divisibility graph $D(G)$? In general, our answer to this question is no. Because we obtained a counterexample, that is the group of the Small Groups library of GAP with number Id(210, 2) has $cs^*(G) = \{2, 3, 5, 6, 7, 14, 15, 35\}$. Therefore the divisibility graph $D(G)$ is 2-regular but it is not a complete graph. In Theorem 2.4, we will provide an affirmative answer to the question for the class of F-groups. After the next proposition we will be ready to show that, given an F-group G, the divisibility graph $D(G)$ is never regular unless it is complete.

Proposition 2.3. *Let G be a nonabelian F-group and for $k \geqslant 1$ the divisibility graph $D(G)$ be a k-regular graph. Then the divisibility graph $D(G)$ is a connected graph.*

Proof. If the divisibility graph $D(G)$ is not connected, then by Theorem 2.2, it contains at least one isolated vertex and so it is not a k-regular graph for $k \geqslant 1$. □

Theorem 2.4. *Let G be a nonabelian F-group. Then, for $k \geqslant 1$, the divisibility graph $D(G)$ is k-regular if and only if it is a complete graph with $k+1$ vertices.*

Proof. We need only to prove the "only if" part. Let G be a nonabelian F-group and the divisibility graph $D(G)$ is k-regular. So by Proposition 2.3, the divisibility graph $D(G)$ is connected and by Theorem 2.2, it is a star. Since the divisibility graph is k-regular, thus the divisibility graph $D(G)$ is a complete graph. □

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