Generic Results for Groups of Lie Type

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Root data

Definition

A pair $(A, A^\vee)$*, with* $A, A^\vee \in \mathbb{Z}^{l\times r}$*, is called a root datum if* $C = A^\vee A^\mathsf{T}$ *is a Cartan matrix of a crystallographic root system.*

Remarks. Let $(A, A^\vee)$ be a root datum.

(a) A root datum determines (up to isomorphism) a connected reductive group G_K over any algebraically closed field K.

(b) Let $X = \mathbb{Z}^r$, α_i the i-th row of A, $\alpha_i^\vee$ the i-th row of $A^\vee$, then

$$s_i : X \to X, \quad x \mapsto x - (\alpha_i^\vee \cdot x^\mathsf{T})\alpha_i$$

is a reflection and

$$W = \langle s_i \mid 1 \leq i \leq l \rangle$$

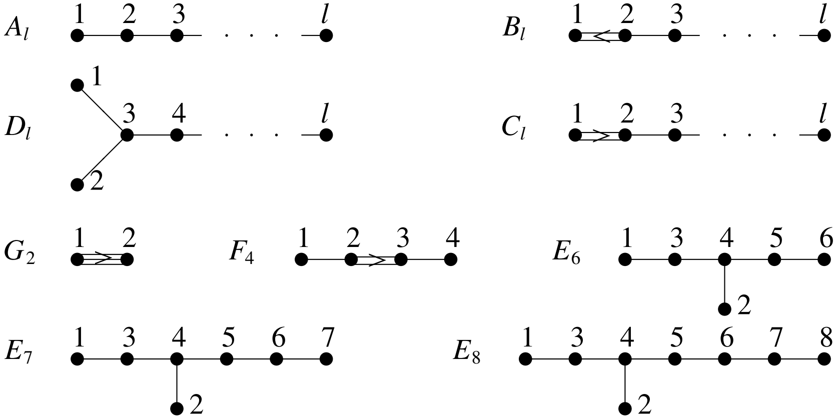
is the Weyl group of $(A, A^\vee)$ and of G_K.

(c) $(K^\times)^r \cong T \leq G_K$, a maximal torus of G_K, and $W \cong N_{G_K}(T)/T$.

(d) The W-orbits of the rows of A are the roots of G_K.

Classification of Cartan matrices

Up to reordering a Cartan matrix $C = (c_{ij})$ is block diagonal with blocks described by:



$c_{ii} = 2$, $c_{ij} = c_{ji} = 0$ if no bond, $c_{ij}c_{ji} = k$ if k bonds, $c_{ij} \in \{0, -1, -2, -3\}$, $c_{ij} < -1$ if arrow from i to j.

Example $\mathsf{GL}_{l+1}(K)$

$(A, A^\vee)$ with

$$A = A^\vee = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & \dots & -1 & 1 \end{pmatrix} \in \mathbb{Z}^{l\times(l+1)},$$

is a root datum.

Here: C is of type A_l, $W \cong S_{l+1} = \langle (1,2),(2,3),\dots(l,l+1)\rangle$, $G_K \cong \mathsf{GL}_{l+1}(K)$, $T =$ diagonal matrices.

Frobenius action

Definition

A triple $(A, A^\vee, F_0)$ is called a root datum with Frobenius action if $A, A^\vee \in \mathbb{Z}^{l\times r}$ form a root datum and $F_0 \in \mathbb{Z}^{r\times r}$ is of finite order and fixes its root system.

Remarks. For each prime p and power $q = p^f$ the matrix F_0 determines a Frobenius morphism $F_q : G_K \to G_K$ in the case $K = \overline{\mathbb{F}_p}$ which defines G_K over $\mathbb{F}_q$.

$G(q) := G_K^{F_q}$ is a finite group.

$\{G(q) \mid q$ a prime power$\}$ is a series of finite groups of Lie type.

Example $\mathsf{GL}_{l+1}(q)$ and $\mathsf{GU}_{l+1}(q)$

Matrices $A, A^\vee \in \mathbb{Z}^{l\times(l+1)}$ as before and $F_0 = E_{l+1}$ yield

$$\{\mathsf{GL}_{l+1}(q) \mid q \text{ a prime power}\}$$

and with $F_0 = -E_{l+1}$ yield

$$\{\mathsf{GU}_{l+1}(q) \mid q \text{ a prime power}\}.$$

Other types: $\mathsf{Sp}_{2l}(q)$, $\mathsf{SO}_{2l+1}(q)$, $\mathsf{Spin}_{2l}^{\pm}(q)$, $G_2(q)$, ${}^2E_6(q)_{sc}$, ...

Example ${}^2E_6(q)_{sc}$

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad A^\vee = E_6,$$

$$F_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

General task

Given a root datum with Frobenius action $(A, A^\vee, F_0)$, compute generic information about the corresponding groups $\{G(q) \mid q\}$ and their representations, that is with q as a parameter.

Examples:

(a) The order $|G(q)|$, like

$$|{}^2E_6(q)_{sc}| = q^{36}(q^4-q^3+q^2-q+1)(q^4+1)(q^4-q^2+1)(q^6-q^3+1)$$
$$(q^2+1)^2(q^2+q+1)^2(q^2-q+1)^3(q-1)^4(q+1)^6,$$

is always a polynomial in q.

(b) The order of the center $|Z(G(q))|$ is always polynomial on residue classes (PORC), e.g.,

$$|Z({}^2E_6(q)_{sc})| = \gcd(3, q+1) = \begin{cases} 3, & \text{if } q \equiv 2 \bmod 3 \\ 1, & \text{if } q \equiv 0 \text{ or } 1 \bmod 3 \end{cases}$$

Conjugacy classes of $G(q)$

Remarks.

(a) (Jordan decomposition) For $g \in G(q)$ there are unique $s, u \in G(q)$ with $g = su = us$ and
s semisimple (diagonalizable, in maximal torus, p'-element)
u unipotent (eigenvalues 1, p-element)

(b) Centralizer $C_G(s)$ is again reductive group (maybe not connected).

(c) Strategy: first find semisimple classes with their centralizers, find unipotent classes in these centralizers.

Semisimple classes

- can be parameterized by computations with root datum and Frobenius action,
- number of classes of centralizers bounded independently of q (described by root data),
- number of classes with fixed type of centralizer is PORC.

Work in progress: parameterize classes of l-elements or l'-elements for $l \neq p$ prime.

Example $G(q) = {}^2E_6(q)_{sc}$

Number of $G(q)$-conjugacy classes of centralizers of semisimple elements is 100 (for $q \equiv 1, 5 \bmod 6$), 92 (for $q \equiv 2, 4 \bmod 6$) or 99 (for $q \equiv 3 \bmod 6$).

Number of semisimple classes of $G(q)$ with centralizer of type $A_2(q^2) + T((q-1)^2)$:

$$\begin{array}{lll} 1/12(q^2 - 8q + 19) & \text{if } q \equiv 1 \bmod 6 & \\ 1/4(q^2 - 8q + 12) & \text{if } q \equiv 2 \bmod 6 & \\ 1/12(q^2 - 8q + 15) & \text{if } q \equiv 3 \bmod 6 & \\ 1/12(q^2 - 8q + 16) & \text{if } q \equiv 4 \bmod 6 & \\ 1/4(q^2 - 8q + 15) & \text{if } q \equiv 5 \bmod 6 & (= 0 \text{ for } q = 5) \end{array}$$

Unipotent classes

- no algorithm known to parameterize unipotent classes directly from root datum,
- (Steinberg) a presentation of G and $G(q)$ can be derived from root datum, useful for direct computations in G,
- unipotent classes worked out for all simple G of simply connected type,
- we can reduce general case via root datum to these cases,
- number of unipotent classes is a PORC constant.

Example $G(q) = {}^2E_6(q)_{sc}$

The algebraic group G always has 21 unipotent classes,
but their intersection with $G(q)$ has different number of classes.

Total number of unipotent classes of $G(q)$ is:

25	if $q \equiv 1 \bmod 6$
44	if $q \equiv 2 \bmod 6$
27	if $q \equiv 3 \bmod 6$
28	if $q \equiv 4 \bmod 6$
39	if $q \equiv 5 \bmod 6$

Their centralizer orders are polynomials in q.

Example $G(q) = {}^2E_6(q)_{sc}$

Finding unipotent classes of all semisimple centralizers, one gets the total number of conjugacy classes of $G(q)$:

$$\begin{array}{ll}
q^6+q^5+2q^4+4q^3+11q^2+11q+16 & \text{if } q \equiv 1 \bmod 6 \\
q^6+q^5+2q^4+4q^3+18q^2+26q+62 & \text{if } q \equiv 2 \bmod 6 \\
q^6+q^5+2q^4+4q^3+11q^2+11q+15 & \text{if } q \equiv 3 \bmod 6 \\
q^6+q^5+2q^4+4q^3+10q^2+10q+14 & \text{if } q \equiv 4 \bmod 6 \\
q^6+q^5+2q^4+4q^3+19q^2+27q+72 & \text{if } q \equiv 5 \bmod 6
\end{array}$$

Generic character tables

Example of character table for $H = A_5 = \langle (1,2,3,4,5), (3,4,5) \rangle \cong SL_2(4)$.

Conjugacy classes: $()^G$, $(1,2)(3,4)^G$, $(1,2,3)^G$, $(1,2,3,4,5)^G$, $(1,2,3,5,4)^G$

Character table (from GAP):

```
          1a 2a 3a 5a 5b

X.1        1  1  1  1  1
X.2        3 -1  .  A *A
X.3        3 -1  . *A  A
X.4        4  .  1 -1 -1
X.5        5  1 -1  .  .
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where $\mathtt{A} = (1+\sqrt{5})/2 = -(\zeta_5^2 + \zeta_5^3)$ and $*\mathtt{A} = (1-\sqrt{5})/2 = -(\zeta_5 + \zeta_5^4)$.

Generic character table of $SL_2(q)$ with $q \equiv 0 \bmod 2$

$SL_2(q)$	C_1	C_2	$C_3(a)$	$C_4(a)$
χ_1	1	1	1	1
χ_2	q	0	1	-1
$\chi_3(n)$	$q+1$	1	$\zeta_1^{an}+\zeta_1^{-an}$	0
$\chi_4(n)$	$q-1$	-1	0	$-\xi_1^{an}-\xi_1^{-an}$

$\zeta_1 := \exp(\frac{2\pi\sqrt{-1}}{q-1})$, $\xi_1 := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

Parameter ranges:

$\chi_3(n)$: $n = 1,\dots,q-2$ ($\frac{1}{2}(q-2)$ characters)
$\chi_4(n)$: $n = 1,\dots,q$ ($\frac{1}{2}q$ characters)
$C_3(a)$: $a = 1,\dots,q-2$ ($\frac{1}{2}(q-2)$ classes)
$C_4(a)$: $a = 1,\dots,q$ ($\frac{1}{2}q$ classes)

Generic table for $SL_2(q)$ with $q \equiv 1 \bmod 2$

$SL_2(q)$	$C_1(i)$	$C_2(i)$
χ_1	1	1
χ_2	q	0
χ_3	$\frac{1}{2}(q+1)(-1)^{\frac{1}{2}(q-1)i}$	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i} + \frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$
χ_4	$\frac{1}{2}(q+1)(-1)^{\frac{1}{2}(q-1)i}$	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i} - \frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$
χ_5	$\frac{1}{2}(q-1)(-1)^{\frac{1}{2}qi+\frac{1}{2}i}$	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i} + \frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$
χ_6	$\frac{1}{2}(q-1)(-1)^{\frac{1}{2}qi+\frac{1}{2}i}$	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i} - \frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$
$\chi_7(k)$	$(q+1)(-1)^{ik}$	$(-1)^{ik}$
$\chi_8(k)$	$(q-1)(-1)^{ik}$	$-(-1)^{ik}$

$\varepsilon_4 := \exp(\frac{2\pi\sqrt{-1}}{4})$, $\quad \zeta_1 := \exp(\frac{2\pi\sqrt{-1}}{q-1})$, $\quad \xi_1 := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

(continued ...)

$SL_2(q)$	$C_3(i)$	$C_4(i)$	$C_5(i)$
χ_1	1	1	1
χ_2	0	1	-1
χ_3	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i} - \frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$	$(-1)^i$	0
χ_4	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i} + \frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$	$(-1)^i$	0
χ_5	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i} - \frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$	0	$-(-1)^i$
χ_6	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i} + \frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$	0	$-(-1)^i$
$\chi_7(k)$	$(-1)^{ik}$	$\zeta_1^{ik} + \zeta_1^{-ik}$	0
$\chi_8(k)$	$-(-1)^{ik}$	0	$-\xi_1^{ik} - \xi_1^{-ik}$

$\varepsilon_4 := \exp(\frac{2\pi\sqrt{-1}}{4})$, $\quad \zeta_1 := \exp(\frac{2\pi\sqrt{-1}}{q-1})$, $\quad \xi_1 := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

Remarks.

(a) $SL_2(q)$ case known to Frobenius, more generic tables by Steinberg (GL_4), Green (GL_n), Srinivasan (Sp_4), ...

(b) Theoretical background for general case by Deligne-Lusztig, Lusztig (1976—).

(c) Parameterization of irreducible characters by Lusztig's Jordan decomposition of characters: involves semisimple classes of dual group (corresponding to $(A^\vee, A, F_0^\top)$) and Lusztig's parameterization of unipotent characters (only depend on root datum, independent of p or q). (in practice very similar to conjugacy classes)

Generic character values

- So far, no general algorithm known to compute full generic character tables except $GL_n(q)$, $GU_n(q)$ (but there is work in progress ...).
- Computation of values involves
  - ”root of unity part” (computable from root datum via semisimple elements)
  - ”polynomial part” which involves generalized Green functions and Lusztig’s theory of character sheaves (computable in many cases, but difficult open cases remain, work in progress ...)
  - certain identification problems
- For many applications partial information is sufficient, and this is sometimes computable, e.g., the character degrees or only the ”smallest” character degrees, certain scalar products of class functions, structure constants for certain tuples of conjugacy classes, ...

Example: generic character table of $E_8(q)$

- Complete computation currently not possible (even table of $E_8(2)$ is not yet known),
- parameterization of conjugacy classes and irreducible characters known (depends on q mod 60),
- for $q \equiv 1$ mod 60 there are
  - there are 876 conjugacy classes of centralizers of semisimple elements,
  - generic character tables has 6345 rows,
  - and 10061 columns,
- ”ordinary” Green functions are now known in all cases (new for small p),
- largest ”root of unity” term has $|W| \sim 7 \cdot 10^8$ summands with 8 parameters for characters and 8 parameters for classes.

Thank You

for your attention!