On character degrees

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joint work with Lorenzo Bonazzi, Gabriel Navarro and Noelia Rizo.

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If $\chi \in \mathrm{Irr}(G)$, then $\chi(1)$ is the **degree** of χ.

We have that $\chi(1)$ divides $|G|$.

Character degrees in Finite Groups is one of the fundamental subjects in Representation Theory

For instance, two of the major problems in representation theory are about character degrees:

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McKay Conjecture

If G is a finite group, p a prime, then

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$$

where $P \in \mathrm{Syl}_p(G)$.

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Brauer's Height Zero Conjecture

Suppose that G is a finite group, p is a prime and B is a p-block of G with defect group D.
All irreducible complex characters in B have height zero if and only if D is abelian.

A classical problem is to find relationships between character degrees and the structure of the group

This is one of the essential results:

Theorem (Itô-Michler)

Let p be a prime. Then p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ if and only if G has an abelian normal Sylow p-subgroup.

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As simple as it looks, it was proved only after the Classification of Finite Simple Groups.

In fact this is one of the first applications of the classification to character theory.

Often the Itô-Michler's Theorem is presented with a dual result by John Thompson:

Theorem

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Let p be a prime. If p divides the non-linear irreducible character degrees of G, then G has a normal p-complement.

This does not use CFSG.

Another related unpublished result due to Navarro and Tiep:

Theorem

Suppose that G is a finite group, and let $p \neq q$ be primes. Then $\mathrm{Irr}_{p'}(G) = \mathrm{Irr}_{q'}(G)$ if and only if there are abelian $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $\mathbf{N}_G(P) = \mathbf{N}_G(Q)$.

where $\mathrm{Irr}_{p'}(G)$ is the set of irreducible characters of p'-degree.

The proof relies on the CFSG.

Our main concern is how to generalize these results and others in a single statement.

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Is this even possible?

Let π be a set of primes.

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Examples :

$$\mathrm{Irr}_{\emptyset}(G) = \{\chi \in \mathrm{Irr}(G) \,|\, \chi(1) = 1\} = \mathrm{Lin}(G)\,.$$

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$$\mathrm{Irr}_\mathbb{P}(G) = \mathrm{Irr}(G).$$

where $\mathbb{P}$ is the set of all the primes.

In all these problems, the main question is to characterize when

$$\mathrm{Irr}_{\pi}(G) = \mathrm{Irr}_{\rho}(G)$$

by group theoretical properties, where π and ρ are sets of primes.

Itô-Michler

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$$\mathrm{Irr}_{p'}(G) = \mathrm{Irr}_{q'}(G)$$

Again: Can these theorems be unified in a single statement?

MAIN IDEA/ TOOL

The McKay conjecture:

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$$

where $P \in \mathrm{Syl}_p(G)$.

Solved for p-solvable groups.

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If p' is replaced by π and G is π-separable, there is also a McKay theorem for Hall π-subgroups by T. Wolf.

PART I. Complex character degrees

G. Navarro, N. Rizo, L. S. *Character degrees in separable groups*

to appear in Proceedings of the American Mathematical Society.

Recall that a group G is π-**separable** if all the composition factors of G are π or π' groups.

The π-separable groups have a unique conjugacy class of π-subgroups of Hall.

(These are the subgroups such that all the prime divisors of $|H|$ are in π and those of $|G : H|$ are in π'.)

Theorem A

Let $\rho \subseteq \pi$ be sets of primes. Suppose that G is a ρ-separable and a π-separable finite group. Then

$$\mathrm{Irr}_{\pi}(G) = \mathrm{Irr}_{\rho}(G)$$

if and only if there exist a Hall π'-subgroup H of G and a Hall ρ'-subgroup K of G such that $\mathbf{N}_G(K) = K'\mathbf{N}_G(H)$ and $K' \cap \mathbf{N}_G(H) = H'$.

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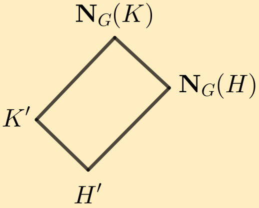

Theorem A

$$\mathrm{Irr}_{\pi}(G) = \mathrm{Irr}_{\rho}(G) \iff \exists H \in \mathrm{Hall}_{\pi'}(G), K \in \mathrm{Hall}_{\rho'}(G) : \mathbf{N}_G(K) = K'\mathbf{N}_G(H), K' \cap \mathbf{N}_G(H) = H'.$$

Itô-Michler

$$\mathrm{Irr}_{p'}(G) = \mathrm{Irr}_{\mathbb{P}}(G) \iff \exists P \in \mathrm{Syl}_p(G) \text{ such that } P \lhd G \text{ and } P' = 1.$$

Set $\rho = p' \subseteq \pi = \mathbb{P}$. Have $H = 1 \in \mathrm{Hall}_{\pi'}(G)$ and $K = P \in \mathrm{Hall}_{\rho'}(G) = \mathrm{Syl}_p(G)$.

Theorem A

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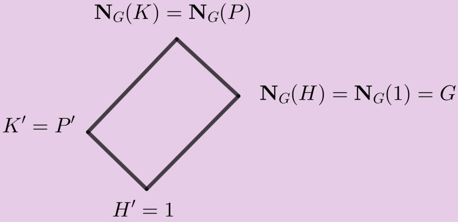

$$\mathbf{N}_G(K) = K'\mathbf{N}_G(H) \Rightarrow \mathrm{N}_G(P) = K'\mathbf{N}_G(1) = G \Rightarrow P \triangleleft G$$
$$K' \cap \mathbf{N}_G(H) = H' \Rightarrow P' \cap \mathbf{N}_G(1) = P' \cap G = P' = 1.$$

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Set $\rho = \emptyset \subseteq \pi = p'$.
Have $H = P \in \mathrm{Hall}_{\pi'}(G) = \mathrm{Syl}_p(G)$ and $K = G$.

$$\mathbf{N}_G(K) = K'\mathbf{N}_G(H) \Rightarrow \mathbf{N}_G(G) = G = G'\mathbf{N}_G(P)$$

$$K' \cap \mathbf{N}_G(H) = H' \Rightarrow G' \cap \mathbf{N}_G(P) = P'.$$

and by Tate's theorem, G has a normal p-complement.

We can also deduce from Theorem A the following results (under the corresponding separability conditions):

Navarro-Tiep, Wolf

Let $p \neq q$ be primes. Then $\mathrm{Irr}_{p'}(G) = \mathrm{Irr}_{q'}(G)$ if and only if there are abelian $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $\mathbf{N}_G(P) = \mathbf{N}_G(Q)$.

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Example:

Let $G = PSL_2(29)$.

The set of irreducible character degrees
$\mathrm{cd}(G) = \{1, 15, 28, 29, 30\} = \{1, 3 \cdot 5, 2^2 \cdot 7, 29, 2 \cdot 3 \cdot 5\}$

Let $\rho = \{29\}$ and $\pi = \{2, 5, 29\}$.

We have that $\mathrm{Irr}_{\rho}(G) = \mathrm{Irr}_{\pi}(G)$ but G does not possess Hall $\{2, 5\}$-subgroups.

Corollary B

Let ρ and π be sets of primes. Suppose that G is a ρ-separable and a π-separable. Then

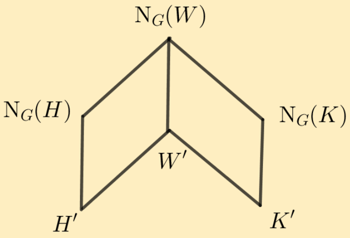
$$\mathrm{Irr}_{\pi}(G) = \mathrm{Irr}_{\rho}(G)$$

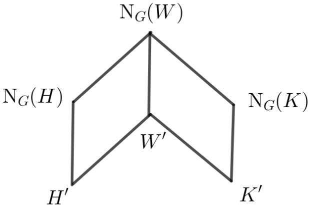
if and only if $\exists H \in \mathrm{Hall}_{\pi'}(G), K \in \mathrm{Hall}_{\rho'}(G)$ *and* $W \in \mathrm{Hall}_{(\pi\cap\rho)'}(G)$ *such that*

$$W'\mathbf{N}_G(H) = \mathbf{N}_G(W) = W'\mathbf{N}_G(K)$$

$$\mathbf{N}_G(H) \cap W' = H' \text{ and}$$

$$W' \cap \mathbf{N}_G(K) = K'.$$





Corollary B

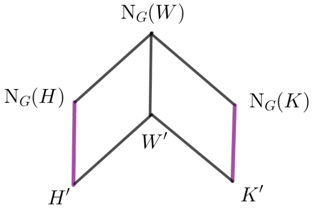
Corollary C

Let ρ and π be sets of primes. Suppose that G is a ρ-separable and π-separable finite group. Suppose that

$$\mathrm{Irr}_{\pi}(G) = \mathrm{Irr}_{\rho}(G)\,.$$

If H is a Hall π'-subgroup of G and K is a Hall ρ'-subgroup of G, then we have that

$$\frac{\mathbf{N}_G(K)}{K'} \cong \frac{\mathbf{N}_G(H)}{H'}\,.$$



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Problem

What does the character table $X(G)$ know about $\frac{\mathbf{N}_G(H)}{H'}$?

PART II. p-Brauer characters degrees

L. Bonazzi, G. Navarro, N. Rizo, L. S. *Brauer Character degrees and Sylow normalizers* (submitted)

Degrees of p- Brauer character outside p-solvable groups have an erratic behavior

Degrees of p- Brauer character outside p-solvable groups have an erratic behavior

The degrees do not necessarily divide the order of the group: A_9 has an irreducible 2-Brauer character of degree 26.

We have a McKay theorem only in p-solvable groups

2-Brauer irreducible characters of A_5

```
        1a  3a  5a  5b
    2P  1a  3a  5b  5a
    3P  1a  1a  5b  5a
    5P  1a  3a  1a  1a

X.1      1   1   1   1
X.2      2  -1   A  *A
X.3      2  -1  *A   A
X.4      4   1  -1  -1

A = E(5)+E(5)^4
  = (-1+Sqrt(5))/2 = b5
```

Let $P \in \mathrm{Syl}_2(G)$. Then $\mathbf{N}_G(P) = A_4$. We have that $|\mathrm{IBr}_{2'}(\mathbf{N}_G(P))| = |\mathrm{IBr}(\mathbf{N}_G(P))| = 3 \neq 1 = |\mathrm{IBr}_{2'}(G)|$

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    3P 1a  1a  5b  5a
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X.1      1   1   1   1
X.2      2  -1   A  *A
X.3      2  -1  *A   A
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- There is no Itô-Michler theorem.

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- There is no Itô-Michler theorem.
- There is no Thompson theorem.

G is p-solvable group

$\mathrm{IBr}(G)$ is the set of irreducible p-Brauer characters of G.

$$\mathrm{IBr}_\pi(G) = \{\chi \in \mathrm{IBr}(G) \,|\, \text{the primes dividing } \chi(1) \text{ are in } \pi\}\,.$$

Theorem D

Let G be a p-solvable group, and let q be a prime. Then

$$\mathrm{IBr}_{q'}(G) \subseteq \mathrm{IBr}_{p'}(G) \text{ if and only if there are}$$

$$Q \in \mathrm{Syl}_q(G) \text{ and } P \in \mathrm{Syl}_p(G) \text{ such that } \mathbf{N}_G(Q) \subseteq \mathbf{N}_G(P).$$

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Beltrán-Navarro proved Theorem D with the hypothesis that G q-solvable

The deepest part of the proof of Theorem D comes from the following result

Theorem (Malle-Navarro)

Let G be a finite π-separable group. Let H be a Hall π-subgroup, let K be a π-complement of G, and let q be a prime.
Then every $\alpha \in \mathrm{Irr}_{q'}(H)$ extends to G if and only if there is $Q \in \mathrm{Syl}_q(H)$ such that $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(K)$.

The deepest part of the proof of Theorem D comes from the following result

Theorem (Malle-Navarro)

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Then every $\alpha \in \mathrm{Irr}_{q'}(H)$ extends to G if and only if there is $Q \in \mathrm{Syl}_q(H)$ such that $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(K)$.

This is an extension of a classical result:
If q does not divide $|H|$,
all characters of a Hall subgroup extend $\Rightarrow$ that H has a normal complement.

Theorem E

Suppose that G is a p-solvable finite group and let q be a prime different from p. Then

$$\mathrm{IBr}_{p'}(G) = \mathrm{IBr}_{q'}(G)$$

if and only if there is a Sylow p-subgroup P of G and a Sylow q-subgroup Q of G, such that $\mathbf{N}_G(P) = P\mathbf{N}_G(Q)$ and Q is abelian.

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Suppose that G is a p-solvable finite group and let q be a prime different from p. Then

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if and only if there is a Sylow p-subgroup P of G and a Sylow q-subgroup Q of G, such that $\mathbf{N}_G(P) = P\mathbf{N}_G(Q)$ and Q is abelian.

Notice that if p does not divide $|G|$, this is the Itô-Michler theorem.

Using Isaacs π-characters, the Glauberman-Isaacs correspondence, and some ad-hoc arguments, it is possible to replace in Theorems D and E p' by π, p-solvable groups by π-separable, and Sylow p-subgroups by Hall π-complements.