Some properties of character graphs

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Figure: Georg Frobenius

Frobenius determinant theorem

- Let a finite group G have elements $g_1, g_2, \ldots, g_n$, and let x_{g_i} be associated with each element of G.
- $X_G := (x_{g_i g_j})_{n\times n}$.
- $\det X_G = \prod_{j=1}^{r} P_j(x_{g_1}, x_{g_2}, \ldots, x_{g_n})^{\deg P_j}$

Applications

- (Feit–Thompson theorem) Every finite group of odd order is solvable.
- (Burnside's theorem) If G is a finite group of order $p^a q^b$ where p and q are prime numbers, and a and b are non-negative integers, then G is solvable.
- (Richard Brauer and Michio Suzuki)[1] A finite simple group cannot have a generalized quaternion group as its Sylow 2-subgroup.

[1] R. Brauer, M. Suzuki, On finite groups of even order whose 2-Sylow group is a quaternion group, Proc. Natl. Acad. Sci. USA, 45 (12) (1959) 1757–1759

Definitions and Notations

Throughout this talk G is a non-abelian finite group. Also

- $\pi(n)$: The set of prime divisors of a positive integer n,
- $\mathrm{Irr}(G)$: The set of all irreducible characters of G,
- $\mathrm{cd}(G) := \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$,
- $\rho(G) := \{p \in \pi(G) \mid p \mid \chi(1), \textit{ for some } \chi \in \mathrm{Irr}(G)\}$.
- $R(G)$: the solvable radical of G.

Suppose Γ is a finite simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$.

- $\chi(\Gamma)$: The chromatic number of Γ,
  Minimum number of colors needed to color vertices of the graph Γ so that any two adjacent vertices of Γ have deferent colors, is called the chromatic number of Γ.
- $\omega(\Gamma)$: The clique number of Γ,
  A clique of Γ is a set of mutually adjacent vertices, and that the maximum size of a clique of Γ is called the clique number of Γ.
- Γ^c: The complement of Γ.
- $\mathrm{diam}(\Gamma)$: The diameter of Γ.

- Let X be a subset of $V(\Gamma)$, the subgraph of Γ whose vertex set is X and whose edge set consists of all edges of Γ which have both ends in X is called the induced subgraph of Γ on X and denoted by $\Gamma[X]$.
- A cut vertex of Γ is a vertex v such that the number of connected component of $\Gamma - v$ is more than the number of connected component of Γ.
- A maximal connected subgraph B of Γ so that B has no any cut vertex is called a block.

Huppert Conjecture

- Huppert Conjecture: Let G be a finite group and S a finite non-abelian simple group such that $\mathrm{cd}(G) = \mathrm{cd}(S)$. Then $G \cong S \times A$, where A is an abelian group.
- Huppert [1]: The conjecture is true for many non-abelian simple groups, including the Suzuki groups, many of the sporadic simple groups, and a few of the simple groups of Lie type.
- Berkovich [2]: If a prime p divides every non-linear character degree of a group G, then G is solvable.

[1] B. Huppert, Some simple groups which are determined by the set of their character degrees I. III., J. Math, 44 (2000) 828–842.

[2] Y. Berkovich, Finite groups with small sums of degrees of some non-linear irreducible characters, J. Algebra, 171 (1995) 426-443.

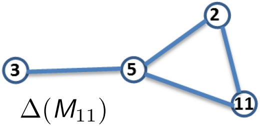
The character graph $\Delta(G)$

$$\Delta(G) := (V_\Delta, E_\Delta)$$

$$V_\Delta := \rho(G)$$

$$E_\Delta := \{e_{p,q} |\ p, q \in \rho(G),\ pq | \chi(1),\ \text{for some } \chi \in \mathrm{Irr}(G)\}$$

This graph was first defined in 1988.[1]



$\Delta(M_{11})$

$$\mathrm{cd}(M_{11}) = \{1, 10, 11, 16, 44, 45, 55\}$$

[1] O. Manz, R. Staszewski, W. Willems, On the number of a components of graph related to character degrees, Proc. Amer. Math. Soc., 103(1) (1988) 31-37.

Character graphs of solvable groups

We assume that G is solvable. Then:

- (1985)[1] $\Delta(G)$ has at most two connected components.
- (1988)[2] Any three primes in $\rho(G)$ must have an edge in $\Delta(G)$ that is incident to two of those primes.
- (1989)[3] $\mathrm{diam}(\Delta(G)) \leq 3$.

[1] O. Manz, Degree problems II: separable character degrees, Comm. Algebra, 13 (1985) 2421-2431.

[2] P.P. Palfy, On the character degree graph of solvable groups I: Three primes, Period. Math. Hungar., 36, (1998) 61-65.

[3] O. Manz, W. Willems, T.R. Wolf, The diameter of the character degree graph, J. Reine Angew. Math., 402 (1989) 181-198.

- The solvable group G is said to be disconnected if $\Delta(G)$ is disconnected.
- (2001)[1] If $\Delta(G)$ has two connected components so that the cardinalities of the vertex sets of these components are n and N with $N \geqslant n$, then $N \geqslant 2^n - 1$.
- (2001)[2] Disconnected groups have been completely classified by Lewis into six types.

[1] P.P. Palfy, On the character degree graph of solvable groups II: Disconnected graphs, Studia Sci. Math. Hungar., 38 (2001) 339-355.

[2] M.L. Lewis, Solvable groups whose degree graphs have two connected components, J. Group Theory, 4(3) (2001) 255-275.

- Let G be a solvable group with disconnected character graph $\Delta(G)$.
  - (1993)[1]: G has Fitting height at most 4, and $G/F(G)$ has derived length at most 4 where $F(G)$ is the Fitting subgroup of G.
  - (2001) [2]: If the connected components of $\Delta(G)$ have at least 2 vertices, then G has Fitting height 3.

  Suppose G is solvable and $\mathrm{diam}(\Delta(G)) = 3$.
- (2016)[3]: There exists a prime p such that $G = PH$, with P a normal non-abelian Sylow p-subgroup of G and H a p-complement.
- (2016)[4]: The group G has Fitting height 3.

[1] O. Manz, T.R Wolf, Representations of solvable groups, Cambridge University Press, Cambridge, 1993.

[2] M.L. Lewis, Solvable groups whose degrees graphs have two connected components, J. Group Theory, 4 (2001) 255-275.

[3] C. Casolo, S. Dolfi, E. Pacifici, L. Sanus, Groups whose character degree graph has diameter three, Israel J. Math, 215 (2016) 523-558.

[4] C.B. Sass, Character degree graphs of solvable groups with diameter three, J. Group Theory, 19 (2016) 1097-1127.

Taketa Problem

For solvable groups, the Taketa problem (also known as the Issacs-Seitz conjecture) is the conjecture for all solvable groups G that $\mathrm{dl}(G) \leqslant |\mathrm{cd}(G)|$.

- (2013)[1]: The Taketa inequality holds when either all the degrees in $\mathrm{cd}(G)$ are odd, when all of the degrees in $\mathrm{cd}(G) - \{1\}$ are even, and when all of the degrees in $\mathrm{cd}(G) - \{1\}$ have the same set of prime divisors.
- (2015)[2]: Let G be a solvable group where $\Delta(G)$ is connected and has diameter three. Then $\mathrm{dl}(G) \leqslant |\mathrm{cd}(G)|$.

[1] K. Aziziheris, M.L. Lewis, Taketa's theorem for some character degree sets, Arch. Math. (Basel), 100(3) (2013) 215-220.

[2] M.L. Lewis, C.B. Sass, The Taketa problem and character degree graphs with diameter three, Algebr Represent Theory, 18 (2015) 1395-1399.

Character graphs of non-solvable groups

Suppose G is a non-solvable group.

- (2007)[1] $\mathrm{diam}(\Delta(G)) \leq 3$.
- (2009)[2] When G is a simple group, then $\mathrm{diam}(\Delta(G)) = 3$ if and only if G is the first Janko's sporadic simple group J_1.
- (2008)[3] $\Delta(G)$ has at most three connected components.

[1] M.L. Lewis, D.L. White, Diameters of degree graphs of non-solvable groups II, J. Algebra, 312(2) (2007) 634-649.

[2] D.L. White, Degree graphs of simple groups, Rocky Mountain J. Math, 39(5) (2009) 1713-1739.

[3] A. Moreto, P.H. Tiep, Prime divisors of character degrees, J. Group Theory, 11 (2008) 341-356.

- (2003)[1] $\Delta(G)$ has three connected components if and only if $G = S \times A$, where $S \cong \mathrm{PSL}_2(2^n)$ for some integer $n \geqslant 2$ and A is an abelian group.
- (2020)[2] If $\omega(\Delta(G)) \geq 5$, then $|\rho(G)| \leq 3\omega(\Delta(G)) - 4$.
- (2019)[3] If $\Delta(G)$ is regular, then $\Delta(G)^c$ is a bipartite graph.

[1] M.L. Lewis, D.L. White, Connectedness of degree graphs of non-solvable groups, J. Algebra, 266(1) (2003) 51-76.

[2] Z. Akhlaghi, S. Dolfi, E. Pacifici, L. Sanus, Bounding the number of vertices in the character degree graph of finite groups, J. Pure Appl. Algebra, 224 (2020) 725-731.

[3] Z. Sayanjali, Z. Akhlaghi, B. Khosravi, On the regularity of character degree graphs, Bull. Aust. Math. Soc., 3(100) (2019) 428-433.

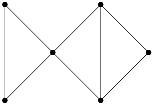
Character graphs with diameter 3

For a finite solvable group G, suppose $\Delta(G)$ has exactly diameter 3. Lewis[1] showed that:

- $\rho(G) = \rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4$ (Lewis partition of $\rho(G)$), where
  - no prime in ρ_1 is adjacent to any prime in $\rho_3 \cup \rho_4$,
  - no prime in ρ_4 is adjacent to any prime in $\rho_1 \cup \rho_2$,
  - every prime in ρ_2 is adjacent to some primes in ρ_3 and vice-versa,
  - $\rho_1 \cup \rho_2$ and $\rho_3 \cup \rho_4$ both determine complete subgraphs of $\Delta(G)$.

[1] M.L. Lewis, Solvable groups with character degree graphs having 5 vertices and diameter 3, Comm. Algebra, 30 (2002) 5485-5503.

- The first Janko's sporadic simple group J_1 is the only well-known non-sovlable group whose character graph has diameter three.



$\Delta(J_1)$

- If G is a finite group with $\mathrm{diam}(\Delta(G)) = 3$, then $\rho(G)$ has a Lewis partition presented at the previous slide[1].

[1] M. Ebrahimi, Character graphs with diameter three, Proc. Amer. Math. Soc., 148(11) (2020) 4615-4619.

Perfect character graphs

- The graph Γ is perfect if $\omega(\Delta) = \chi(\Delta)$, for every induced subgraph Δ of Γ.
- (2006)[1]. A graph Γ is perfect if and only if it has no induced subgraph isomorphic either to a cycle of odd order at least 5, or to the complement of such a cycle.
- For a finite group G, the graph $\Delta(G)$ is a perfect graph.[2]
- $\chi(\Delta(G)^c) \leq 3$.[2]

[1] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, The strong perfect graph theorem, Ann. of Math., (2), 164(1)(2006) 51-229.

[2] M. Ebrahimi, The character graph of a finite group is perfect, Bull. Aust. Math. Soc.,(2020) DOI: 10.1017/S0004972720001240.

Let G be a finite group.

- $\pi \subseteq \rho(G)$ and $|\pi|$ is an odd number.
- (2019)[1] $\Delta(G)^c[\pi]$ is a cycle if and only if $O^{\pi'}(G) = S \times A$, where A is abelian, $S \cong \mathrm{SL}_2(u^\alpha)$ or $S \cong \mathrm{PSL}_2(u^\alpha)$ and the primes in $\pi - \{u\}$ are alternately odd divisors of $u^\alpha \pm 1$.

[1] Z. Akhlaghi, C. Casolo, S. Dolfi, E. pacifici, L. Sanus, On the character degree graph of finite groups, Ann. Mat. 198(5) (2019) 1595-1614.

Hamiltonian cycles

- A graph Γ with n vertices is called Hamiltonian if it contains a cycle of length n.
- (2015)[1] $\Delta(G)$ is Hamiltonian if and only if $\Delta(G)$ is a block with at least 3 vertices.
- (2020)[2] $\Delta(G)^c$ is a non-bipartite Hamiltonian graph if and only if $G \cong \mathrm{SL}_2(2^f) \times A$, where $f \geqslant 2$ is an integer,

$$||\pi(2^f + 1)| - |\pi(2^f - 1)|| \leqslant 1$$

and A is an abelian group.

[1] M. Ebrahimi, A. Iranmanesh, M.A. Hosseinzadeh, Hamiltonian character graphs, J. Algebra, 428 (2015) 54-66.

[2] M. Ebrahimi, Character graphs with non-bipartite Hamiltonian complement., Bull. Aust. Math. Soc., 102 (2020) 91-95.

Dominating sets

- A dominating set for a graph Γ with vertex set V is a subset D of V such that every vertex not in D is adjacent to at least one member of D.
- The domination number of Γ is the number of vertices in a smallest dominating set for Γ.
- If D is a dominating set of Γ such that each $x \in D$ is contained in the set of vertices of an odd cycle of Γ, then we say that D is an odd dominating set of Γ.

Let G be a finite group. Then the following are equivalent:[1]

- The complement of $\Delta(G)$ contains an odd dominating set D.
- The complement of $\Delta(G)$ is non-bipartite with domination number 1.
- $\Delta(G)$ is a disconnected graph with non-bipartite complement.

[1] M. Ebrahimi, Disconnected character graphs and odd dominating sets, Comm. Algebra, 49(8) (2021) 3310-3314.

K_n-free character graphs

Let G be a finite group

- (conjecture)[1] Suppose $\Delta(G)$ is K_n-free. If G is solvable, then $|\rho(G)| \leqslant 2n-2$, and if G is non-solvable, then $|\rho(G)| \leqslant 2n-1$.
- (2018)[2] If G is solvable, then the complement of $\Delta(G)$ is bipartite.
- $G := \mathrm{PSL}_2(61) \times \mathrm{PSL}_2(67) \times \mathrm{PSL}_2(83) \times \mathrm{PSL}_2(157)$.

[1] Z. Akhlaghi, H.P. Tong-Viet, Finite groups with K_4-free prime graphs, Algebr. Represent. Theory, 18(1) (2015) 235-256.

[2] Z. Akhlaghi, C. Casolo, S. Dolfi, K. Khedri, E. Pacifici, On the character degree graph of solvable groups, Proc. Amer. Math. Soc., 146 (2018) 1505-1513.

- If $\Delta(G)$ is K_4-free, then $|\rho(G)| \leq 7$.[1]
- If $|\rho(G)| = 7$, then for some integer $f \geqslant 2$, $G \cong \mathrm{PSL}_2(2^f) \times R(G)$, where $|\pi(2^f \pm 1)| = 1, 2$ or 3.[2]
- If $\Delta(G)$ is K_5-free, then $|\rho(G)| \leq 9$.[3]

[1] Z. Akhlaghi, H.P. Tong-Viet, Finite groups with K_4-free prime graphs, Algebr. Represent. Theory, 18(1) (2015) 235-256.

[2] M. Ebrahimi, K_4-free character graphs with seven vertices, comm. Algebra, 48(3) (2020) 1001-1010.

[3] Z. Akhlaghi, K. Khedri, B. Taeri, Finite groups with K_5-free prime graphs, Commun. Algebra, 47(7) (2019) 2577-2603.

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- If for some integer $n \geqslant 4$, Γ is a K_n-free graph whose complement has an odd cycle of length at least $2n-5$, then we say that Γ is an n-exact graph.
- A planar graph whose complement has a cycle of length 5 is 5-exact.
- A tree with $2n+1$ leaves is $(n+3)$-exact, where n is a positive integer.
- K_4-free graphs with non-bipartite complement are 4-exact.

Let G be a finite group, and $n \geqslant 4$ be an integer.

- If $\Delta(G)$ is an n-exact graph, then $|\rho(G)| \leqslant 2n-1$.[1]
- If $|\rho(G)| = 2n-1$, then for some integer $\alpha \geqslant 2$, $G \cong \mathrm{PSL}_2(2^\alpha) \times R(G)$, where $|\pi(2^\alpha \pm 1)| = n-3, n-2$ or $n-1$. Also

  i) If $|\pi(2^\alpha \pm 1)| = n-3$, then for some disconnected groups A and B of disconnected Types 1 or 4, $R(G) \cong A \times B$, $|\rho(A)| = |\rho(B)| = 2$ and $\rho(G) = \pi(S) \uplus \rho(A) \uplus \rho(B)$.
  ii) If $|\pi(2^\alpha \pm 1)| = n-2$, then $R(G)$ is a disconnected group of disconnected Type 1 or 4, $|\rho(R(G))| = 2$ and $\rho(G) = \pi(S) \uplus \rho(R(G))$.
  iii) If $|\pi(2^\alpha \pm 1)| = n-1$, then $R(G)$ is abelian and $\rho(G) = \pi(S)$.

[1] M. Ebrahimi, *n*-exact character graphs, Cumm. Algebra, (2022), DOI: 10.1080/00927972.2022.2026373.

Regular character graphs

- For some positive integer n, the complement of the disjoint union of n copies of K_2 is called the cocktail party graph $\mathrm{cp}(n)$.
- (Conjecture)[1] Let G be a group. If $\Delta(G)$ is k-regular, for some integer $k \geqslant 2$, then $\Delta(G)$ is either a complete graph of order $k+1$ or a cocktail party graph of order $k+2$.
- $\Delta(G)$ is 3-regular if and only if $\Delta(G) \cong K_4$.[1]

[1] H.P. Tong-Viet, Finite groups whose prime graphs are regular, J. Algebra, 397 (2014) 18-31.

- For a solvable group G, if $\Delta(G)$ is regular with n vertices, then $\Delta(G)$ is either complete or $(n-2)$-regular.[1]
- A regular character-graph $\Delta(G)$ with odd order, of a group G, is a complete graph.[2]
- If $\Delta(G)$ is a k-regular character-graph whose eigenvalues are in the interval $[-2, \infty)$. Then the conjecture is true.[3]

[1] C.P. Morresi Zuccari, Regular character degree graphs, J. Algebra, 411 (2014) 215-224.

[2] Z. Sayanjali, Z. Akhlagi, and B. Khosravi, On the regularity of character degree graphs, Bull. Aust. Math. Soc., 100(3) (2019) 428-433.

[3] M. Ebrahimi, M. Khatami, Z. Mirzaei, Regular character-graphs whose eigenvalues are greater than or equal to -2, Submitted.


Thanks for your attention