Bounds for the number of p'-degree characters of finite groups and their p-blocks

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Overview

This talk has two main parts:

Part 1: We present some known bounds for the conjugacy class number of finite groups.

Part 2: We present some bounds on the number of representations/characters of certain degrees and/or values. These bounds naturally arise from conjugacy class number bounds and the so-called local/global conjectures (McKay, McKay-Navarro, Alperin-McKay,...).

Bounding $k(G)$ in terms of $|G|$

- Let $k(G)$ be the number of conjugacy classes of G.

Theorem (Landau, 1903)

For any positive integer k, there are finitely many isomorphism classes of finite groups with exactly k conjugacy classes.

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Theorem (Pyber, 1992)

There exists an explicitly computable constant $\epsilon > 0$ such that every group G of order n:

$$k(G) > \epsilon \frac{\log n}{(\log \log n)^8}.$$

Bounding $k(G)$ in terms of p

Theorem (Brauer, 1942)

Let $P \in \mathrm{Syl}_p(G)$ with $|P| = p$. Then the principal p-block B_0 of G has $e + \frac{p-1}{e}$ ordinary irreducible characters, where $e := |\mathbf{N}_G(P) : \mathbf{C}_G(P)|$.

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- Brauer's result implies that $k(G) \geqslant 2\sqrt{p-1}$ when $|G|$ is divisible by p but not by p^2. This later was conjectured to be true for all groups of order divisible by p.

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Theorem (Maróti, 2016)

Let G be a finite group of order divisible by p. Then $k(G) \geqslant 2\sqrt{p-1}$.

This was proved earlier for solvable groups by Héthelyi and Külshammer (2000) and by Malle for non-p-solvable groups by Malle (2006).

Conjecture (McKay, 1972)

If G is a finite group, p is a prime, and $\mathrm{Irr}_{p'}(G)$ is the set of all irreducible characters of G of degree not divisible by p, then

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|,$$

where P is a Sylow p-subgroup of G.

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- Example: The alternating group $G = \mathsf{A}_5$ has irreducible characters of degrees $1, 3, 3, 4, 5$. Let $p = 5$. The normalizer $\mathbf{N}_G(P) = D_{10}$ has irreducible characters of degrees $1, 1, 2, 2$. So $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))| = 4$.

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- In 2007, Isaacs, Malle, and Navarro reduced the conjecture to a problem on finite simple groups.
- In 2016, Malle and Spath proved the conjecture for the prime $p = 2$.

BOUNDING $|\mathrm{Irr}_{p'}(G)|$

- The McKay conjecture and the bound $k(G) \geqslant 2\sqrt{p-1}$ imply that

$$|\mathrm{Irr}_{p'}(G)| \geqslant 2\sqrt{p-1}$$

for every G of order divisible by p. This was confirmed by Malle and Maróti in 2016.

The McKay-Navarro conjecture

- Let $\mathcal{G}_p$ be the subgroup of the Galois group $\mathcal{G} := Gal(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q})$ consisting of those automorphism $\sigma \in \mathcal{G}$ such that every root of unity $\xi \in \mathbb{Q}(e^{2\pi i/|G|})$ of order not divisible by p to ξ^{p^f} for some $f \in \mathbb{Z}^{\geqslant 0}$.

Conjecture (Navarro, 2004)

Let G be a finite group, p a prime, and P a Sylow p-subgroup of G. Then there exists a bijection from $\mathrm{Irr}_{p'}(G)$ to $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ that commute with the action of $\mathcal{G}_p$.

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- The *p*-rationality level of a character χ is $\log_p(c(\mathbb{Q}(\chi))_p)$, where $c(F)$ – the conductor of F – is the smallest positive integer n such that $F \subseteq \mathbb{Q}(e^{2i\pi/n})$.

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- The p-rationality level of a character χ is $\log_p(c(\mathbb{Q}(\chi))_p)$, where $c(F)$ – the conductor of F – is the smallest positive integer n such that $F \subseteq \mathbb{Q}(e^{2i\pi/n})$.

- The numbers of characters in $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ at each p-rationality level are the same.

Bounding $|\mathrm{Irr}_{p',almost\ p-rational}(G)|$

- A character χ is called almost p-rational if its p-rationality level is either 0 or 1.
- The McKay-Navarro conjecture and the bound $k(G) \geqslant 2\sqrt{p-1}$ imply that $|\mathrm{Irr}_{p',almost\ p-rational}(G)| \geqslant 2\sqrt{p-1}$.

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Theorem (H.-Malle-Maróti, 2021)

Let G be a finite group of order divisible by p. Then

$$|\mathrm{Irr}_{p',almost\ p-rational}(G)| \geqslant 2\sqrt{p-1}.$$

Moreover, $|\mathrm{Irr}_{p',almost\ p-rational}(G)| = 2\sqrt{p-1}$ iff $|\mathrm{Irr}_{p',almost\ p-rational}(\mathbf{N}_G(P))| = 2\sqrt{p-1}$ (which happens when P is cyclic and $\mathbf{N}_G(P)$ is isomorphic to the Frobenius group $P \rtimes C_{\sqrt{p-1}}$).

The Alperin-McKay conjecture

▸ For a p-block B of a finite group, let $k_0(B)$ denote the number of height zero characters of B.

Conjecture (Alperin, 1975)

Let B be a block of G. Then $k_0(B) = k_0(b)$, where b, a block of $\mathbf{N}_G(P)$, is the Brauer correspondent of B.

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- For principal blocks, we would have $k_0(B_0(G)) = k_0(B_0(\mathbf{N}_G(P)) = k(\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P)P')$, and therefore $k_0(B_0(G)) \geqslant 2\sqrt{p-1}$.

Height 0 characters in blocks

Theorem (H.-Schaeffer Fry-Vallejo, 2021)

Let G be a finite group of order divisible by p. Then the number of p'-degree characters in the principal block of G is always at least $2\sqrt{p-1}$. In other words,

$$p \leqslant \frac{1}{4} k_0(B_0(G))^2 + 1.$$

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- This confirms the principal block case of a conjecture of Héthelyi and B. Külshammer in 2000 that $k(B) \geqslant 2\sqrt{p-1}$ for every block B of positive defect.

Height 0 characters in blocks

Theorem (H.-Schaeffer Fry-Vallejo, 2021)

Let G a finite group and p a prime. Let P be a Sylow p-subgroup and B_0 denote the principal p-block of G. We have:

1) For $k \in \{2, 3\}$, $k_0(B_0) = k$ if, and only if, P has order k.

2) $k_0(B_0) = 4$ if, and only if, exactly one of the following happens:

(i) $[P : P'] = 4$,

(ii) $|P| = 5$ and $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] = 2$.

3) $k_0(B_0) = 5$ if, and only if, exactly one of the following happens:

(i) $|P| = 5$ and $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{1, 4\}$,

(ii) $|P| = 7$ and $[\mathbf{N}_G(P) : \mathbf{C}_G(P)] \in \{2, 3\}$.

The cases $k_0(B_0) = 3$ for $p = 3$ and $k_0(B_0) = 4$ for $p = 2$ were proved by Navarro, Sambale, and Tiep in 2018.

- Brauer's Problem 21 predicts that, for every positive integer k, there are finitely many isomorphism classes of groups which can occur as defect groups of blocks with k ordinary irreducible characters. This was shown by to be a consequence of the Alperin-McKay conjecture and Zelmanov's solution of the restricted Burnside problem by Külshammer and Robinson. We propose the following variation of Brauer's problem 21 for height zero characters.

Conjecture (H.-Schaeffer Fry-Vallejo, 2021)

For every positive integer k_0, there are finitely many isomorphism classes of (abelian) groups (of prime power order) which can occur as abelianizations of defect groups of blocks (of finite groups) with precisely k_0 height-zero irreducible characters.

Thank you very much for your attention!