Finite groups and orders of their elements

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The spectrum and prime graph

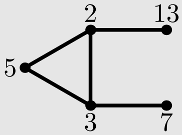
Let G be a finite group. We work with the following objects:

- $\pi(G)$, the set of prime divisors of the order of G
- $\omega(G)$, the spectrum, the set of element orders of G
- $\mu(G)$, the set of maximal under divisibility elements of $\omega(G)$
- $GK(G)$, the prime graph (or Gruenberg–Kegel graph) of G: the vertex set of this graph is $\pi(G)$ and $p, q \in \pi(G)$ are adjacent iff $p \neq q$ and $pq \in \omega(G)$.

Example: $L = U_4(5) = PSU_4(5)$

$\pi(L) = \{2, 3, 5, 7, 13\}$
$\omega(L) =$ divisors of $\mu(L)$
$\mu(L) = \{24, 52, 60, 63\}$

$GK(L)$:



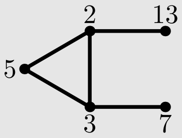
Independence numbers of a prime graph

- $s(G)$, the number of connected components of $GK(G)$
- $t(G)$, the independence number of $GK(G)$, the maximal number of pairwise nonadjacent vertices of $GK(G)$
- $t(r, G)$, the r-independence number of $GK(G)$, the maximal number of pairwise nonadjacent vertices of $GK(G)$ including $r \in \pi(G)$.

Example: $L = U_4(5) = PSU_4(5)$

$\pi(L) = \{2, 3, 5, 7, 13\}$
$\mu(L) = \{24, 52, 60, 63\}$
$s(L) = 1,\ t(L) = 3,\ t(2, L) = 2$

$GK(L)$:



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Higman (1957): $t(G) > 2 \Rightarrow G$ is nonsolvable.

Gruenberg and Kegel (Williams, 1981): $s(G) > 1 \Rightarrow \text{ncf}(G) \leq 1$.

Vasil'ev (2005): $t(2, G) > 1 \Rightarrow \text{ncf}(G) \leq 1$.

- $\text{ncf}(G)$, the number of nonabelian composition factors of G.

Spectra and prime graphs of simple groups

For every finite nonabelian simple group L, we know

- $s(L)$ and the components (Williams 1981, Kondrat'ev 1989)
- $GK(L)$, $t(L)$, and $t(2, L)$ (Vasil'ev – Vdovin 2005)
- $\omega(L)$ (…Buturlakin 2018).

Properties of prime graphs of nonabelian simple groups

1. If L is sporadic or exceptional of Lie type, not $E_7(q)$, then $s(L) > 1$.
2. If L is a nonabelian simple group, not alternating, then $t(r, L) > 1$ for each $r \in \pi(L)$, in particular, $t(2, L) > 1$.
3. If L is a nonabelian simple group, then either $s(L) > 1$, or $t(L) > 2$, or $L = A_{10}$.

Isospectral groups

- Groups G and H are isospectral if $\omega(G) = \omega(H)$
- $h(G)$, the number of groups isospectral to G.

Theorem (Shi and Mazurov, 1998...2012)

1. If G has a normal elementary abelian subgroup $N \neq 1$, then $\omega(G) = \omega(N \rtimes G) = \omega(N \rtimes (N \rtimes G)) = \dots$, so $h(G) = \infty$.
2. If $h(G) = \infty$, then there is a group H with $\omega(H) = \omega(G)$ having a nontrivial normal elementary abelian subgroup.

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Let G be a group isospectral to some nonabelian simple group L. What can we say about the structure of G?

M. A. Grechkoseeva, V. D. Mazurov, W. Shi, A. V. Vasil'ev, and N. Yang, *Finite groups isospectral to simple groups*, accepted to Comm. Math. Stat.; arXiv:2111.15198 [math.GR].

General Structure of a Group Isospectral to Simple

L is nonabelian simple and G is a group with $\omega(G) = \omega(L)$.

Williams (+ Gruenber – Kegel's theorem) 1981, Kondrat'ev 1989, Vasil'ev and Vasil'ev – Vdovin 2005, Gorshkov 2012:

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Theorem (Lucido and Moghaddamfar 2004)

If G is solvable, then $L \in \{L_3(3), U_3(3), S_4(3), A_{10}\}$.

Theorem (Staroletov 2008)

If $\omega(G) = \omega(A_{10})$, then G is not solvable.

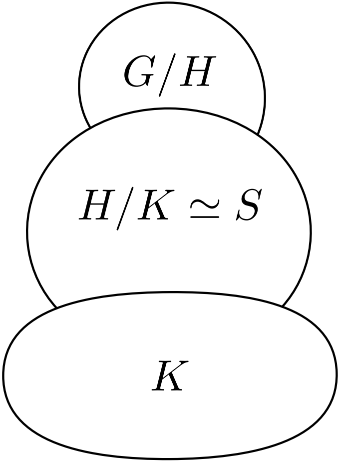
Mazurov (1998, 2002) and Zavarnitsine (2010) found solvable groups isospectral to $L_3(3)$, $U_3(3)$, and $S_4(3)$ respectively.

If $L \neq L_3(3), U_3(3), S_4(3)$, then $\mathrm{ncf}(G) = 1$.

L is nonabelian simple, G is nonsolvable and $\omega(G) = \omega(L)$

$$1 \leq K < H \leq G, \qquad (*)$$

K is the solvable radical, $H/K \simeq S$ is simple, $G/H \leq \mathrm{Out}(S)$.



Theorem (Yang, Grechkoseeva, Vasil'ev, 2020)

If $L \neq A_{10}$ is nonabelian simple, G is nonsolvable and $\omega(G) = \omega(L)$, then the solvable radical K of G is nilpotent.

Mazurov (1998) showed that A_{10} is isospectral to a group of the form $(7^4 \times 3^{12}) : (2.S_5)$, the solvable radical here is Frobenius.

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Theorem (Grechkoseeva and Vasil'ev 2022)

Suppose that a finite group G has a normal series $(*)$, the solvable radical K is nilpotent, $S = H/K$ is a finite simple group of Lie type, $G/K \leq \mathrm{Aut}\, S$. If G/H is not cyclic, then some $r \in \pi(G/H)$ is adjacent to all other primes in $GK(G)$.

Finite groups isospectral to simple

Main Theorem (current version)

Suppose that L is one of the following simple groups:

- sporadic groups, not J_2
- alternating groups, not A_6, A_{10}
- exceptional groups of Lie type, not ${}^3D_4(2)$
- classical groups of dimension $n \geq 37$.

If $\omega(G) = \omega(L)$, then $L \leq G \leq \operatorname{Aut} L$; all such groups G are known.

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In details, see the survey [GMSVY22, Theorem 1 and Tables 1–9].

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Suppose that L is one of the following simple groups:

- sporadic groups, not J_2
- alternating groups, not A_6, A_{10}
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- $L_n(q)$, $U_n(q)$ with $n \geqslant 2$, not $L_2(9)$, $L_3(3)$, $U_3(q)^*$, $U_4(2)$, $U_5(2)$
- $S_{2n}(q)$, $O_{2n+1}(q)$ with $n \geqslant 2$, not $S_4(q)^{**}$, $S_8(q)^{***}$, $O_9(q)$
- $O^+_{2n}(q)$ with $n \geqslant 4$
- $O^-_{2n}(q)$ with $n \geqslant 4$.

Then $h(L) < \infty$. Furthermore, if $\omega(G) = \omega(L)$, then either $h(L) = 2$ and $L, G \in \{S_6(2), O_8^+(2)\}, \{O_7(3), O_8^+(3)\}$, or $L \leq G \leq \operatorname{Aut} L$, and all such groups G are known.

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* In fact, 'not $U_3(q)$, where q is a Mersenne prime such that $q^2 - q + 1$ is prime'. The only known such q are 3 and 7.
** $h(S_4(q)) = 1$ for $q = 3^{2k+1} > 3$ and $h(S_4(q)) = \infty$ otherwise.
*** $h(S_8(7^k)) =$ **?** and $h(S_8(q)) = \infty$ otherwise.

How to make a dream come true?

Prove the following conjecture

Let q be odd and L be one of the following simple groups:

1. $L_n(q)$, where $6 \leqslant n \leqslant 26$ and n is not prime;
2. $U_n(q)$, where $5 \leqslant n \leqslant 26$;
3. $O^+_{2n}(q)$, where $5 \leqslant n \leqslant 18$;
4. $O^-_{2n}(q)$, where $5 \leqslant n \leqslant 17$ and $n \neq 8, 16$;
5. $S_{2n}(q)$ and $O_{2n+1}(q)$, where $5 \leqslant n \leqslant 15$ and $n \neq 8$.

If G is a finite group isospectral to L, then the unique nonabelian composition factor S of G cannot be a group of Lie type in characteristic coprime to q.

Nonsimple groups

If the solvable radical K of G is nontrivial, then $h(G) = \infty$.
So if $h(G) < \infty$, then for some nonabelian simple $L_1, \ldots, L_k$:

$$L_1 \times L_2 \times \cdots \times L_k = \mathrm{Soc}(G) \leq G \leq \mathrm{Aut}(L_1 \times L_2 \times \cdots \times L_k) \quad (**)$$

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If $k = 1$, then G is almost simple. See [GMSVY22, Sect. 4.2].

Example

Let q be a power of a prime p, n an odd number such that $n-1$ is not a power of p and $G = PGL_n(q)$. Then $\omega(G) = \omega(SL_n(q))$. In particular, if $(n, q-1) \neq 1$, then $h(G) = \infty$.

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We know $\omega(L)$ for every simple group L (… Buturlakin 2018).

Dream: $\omega(G)$ for every almost simple group G
(close to come true … Grechkoseeva, Buturlakin).

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Theorem (GMSVY, 2022)

There is an infinite sequence of primes $r_i, i \in \mathbb{N}$, such that $h(G_k) = 1$ for every $k \in \mathbb{N}$, where

$$G_k = \prod_{i=1}^{k} Sz(2^{r_i}).$$

$Sz(q) = {}^2B_2(q)$, $q = 2^{2m+1} > 2$ is the simple Suzuki group.

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If $G = \mathrm{Soc}(G)$, then there is k (depends on L) s.t. $h(G) = \infty$.
If $k \geq |\mu(L)|$, then $\mu(G) = \{m\}$, so $\omega(G) = \omega(\mathbb{Z}_m)$.

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Mazurov (1997): If $G = Sz(2^7) \wr (\mathbb{Z}_{23} : \mathbb{Z}_{11})$, then $h(G) = 1$, though $|\mu(Sz(2^7))| = 4$ and $k = 23$.

Squares of simple groups

All known results

- $h(Sz(2^7) \times Sz(2^7)) = 1$ (Mazurov 1997)
- $h(J_4 \times J_4) = 1$ (Gorshkov and Maslova 2020).

The reason for the lack of results is that the standard methods of proving the nonsolvability of groups isospectral to a simple group are based on properties of its prime graph. However, $GK(L \times L)$ is a complete graph for any group L.

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Theorem (Wang, Vasil'ev, Grechkoseeva, Zhurtov 2022)

Let $L = Sz(q)$, $q \geq 8$, and $\omega(G) = \omega(L \times L)$, then $G \simeq L \times L$.

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Observe that $t(L) = 4$. So we apply

Theorem

Let L be a group with $t(L) \geqslant 4$. If $\omega(G) = \omega(L \times L)$, then G is nonsolvable.

Sufficient criterion of nonsolvability

Theorem (Wang, Vasil'ev, Grechkoseeva, Zhurtov 2022)

Suppose $\pi(G) \supseteq \sigma(G) = \{p_i \mid i \in I\}$, $|I| \geqslant 4$, satisfying:

1. $p_i p_j \in \omega(G)$ for all distinct $i, j \in I$;
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2. If $|I| \geq 6$, then Theorem is true even without Condition 1: (J. Zhang 1995) if G is solvable, then

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3. Theorem is true if p_i does not also divide $p_j - 1$ for all $i, j \in I$ (Gorshkov and Maslova 2020).

Open problems

1. Is it true that for every n there is a group G with $h(G) = 1$ that is the nth direct power of a nonabelian simple group?
2. Is it true that there is a nonabelian simple group L such that for every n, there is a group G with $h(G) = 1$ whose socle is the kth power of L for some $k \geq n$?
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These and other intriguing problems you can find in the new 20th edition of the *Kourovka notebook*, which will appear very soon at https://kourovka-notebook.org/.

You are also welcomed to the *Kourovka Forum*, an additional feature of the *Kourovka Notebook*, the series of online seminars. This spring the invited speakers are A. Lubotzky, D. Segal, R. Solomon, L. Pyber, B. Oliver, and Ch. Parker. See details at http://mca.nsu.ru/kourovkaforum/.