The soluble graph of a finite group

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The commuting graph

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Definition. Let $\Gamma(G)$ be the graph with vertices $G \setminus Z(G)$, where $x \sim y$ if and only if $xy = yx$.

This is the **commuting graph** of G.

Connectedness

Example. Let $G = S_p$, where $p \geqslant 3$ is a prime.

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Theorem (Morgan & Parker, 2013).
If $Z(G) = 1$, then $\mathrm{diam}(\Gamma(G)) \leqslant 10$.

- It is not known if the upper bound of 10 is best possible, but there are examples with $\mathrm{diam}(\Gamma(G)) = 8$.
- **Parker, 2013:** If G is **soluble** and $Z(G) = 1$, then $\mathrm{diam}(\Gamma(G)) \leqslant 8$.

The soluble graph

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Let G be a finite **insoluble** group with soluble radical $R(G)$.

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Definition. Let $\Gamma_{\mathcal{S}}(G)$ be the graph with vertices $G \setminus R(G)$, where $x \sim y$ if and only if $\langle x, y \rangle$ is soluble.

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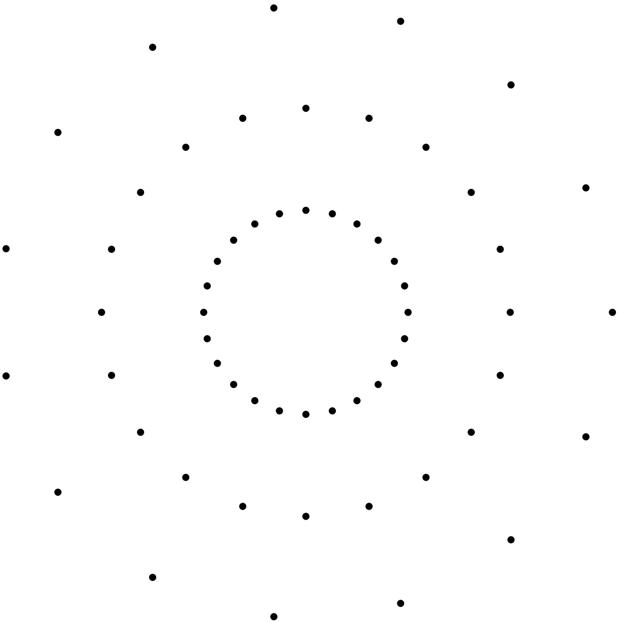
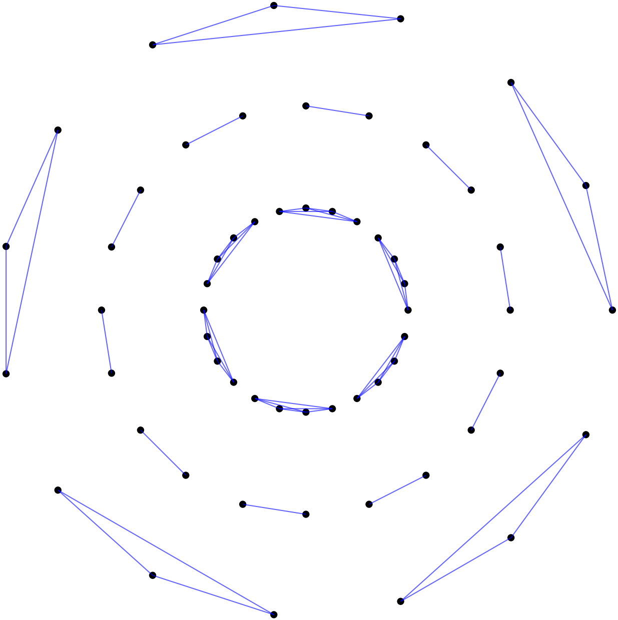
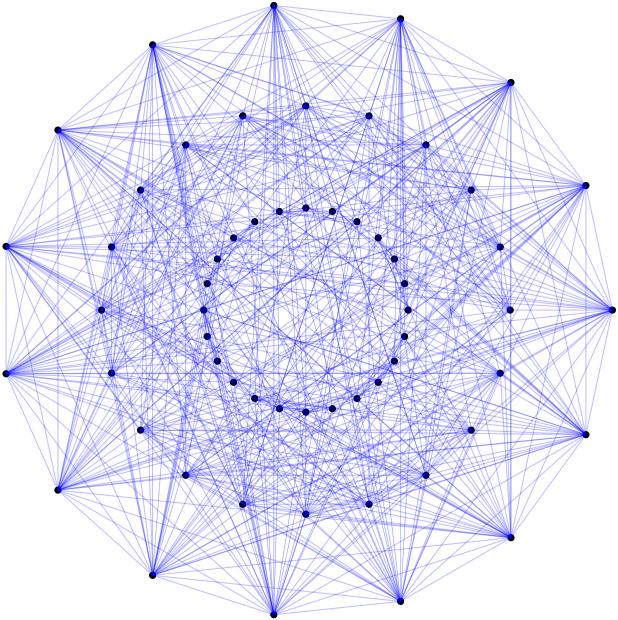
Example: $G = A_5$



Commuting graph of A_5



Soluble graph of A_5



Main results

Let G be a finite insoluble group and let $d_{\mathcal{S}}(G)$ be the maximal diameter of a connected component of the soluble graph $\Gamma_{\mathcal{S}}(G)$.

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Theorem (B, Lucchini & Nemmi, 2021).
$\Gamma_{\mathcal{S}}(G)$ is connected and $d_{\mathcal{S}}(G) \leqslant 5$.

- There are groups with $d_{\mathcal{S}}(G) = 4$, e.g.

$$A_{11},\ A_{12},\ \mathrm{PSL}_5(2),\ \mathrm{PSU}_5(2),\ \mathrm{M}_{12},\ \mathrm{M}_{22},\ \mathrm{M}_{23},\ \mathrm{M}_{24},\ \mathrm{HS},\ \mathrm{J}_3, \dots$$

- M_{12} is the smallest group with $d_{\mathcal{S}}(G) \geqslant 4$.

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- M_{12} is the smallest group with $d_{\mathcal{S}}(G) \geqslant 4$.
- $d_{\mathcal{S}}(G) \geqslant 4 \implies G$ is **almost simple** (i.e. $T \trianglelefteqslant G \leqslant \mathrm{Aut}(T)$)
- There are infinitely many simple groups with $d_{\mathcal{S}}(G) = 2$.
  e.g. $G = \mathrm{PSL}_2(q)$ with $q \geqslant 4$ even.

Earlier work

Theorem (Akbari, Lewis, Mirzajani & Moghaddamfar, 2020).
$\Gamma_{\mathcal{S}}(G)$ is connected and $d_{\mathcal{S}}(G) \leqslant 11$.

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Suppose $R(G) = 1$ and let $\mathrm{Inv}(G)$ be the set of involutions in G.

ALMM: $d(x, \mathrm{Inv}(G)) \leqslant 5$ for all $1 \neq x \in G$, so $d_{\mathcal{S}}(G) \leqslant 11$.

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The following result is a key ingredient in **ALMM** (it relies on CFSG):

Hagie (2000): $\Pi_s(G)$ is connected and $d(2, p) \leqslant 3$ for any prime p.

Involution distance

Our approach yields the following result.

Theorem. If $R(G) = 1$ and $1 \neq x \in G$, then either

- $d(x, \mathrm{Inv}(G)) \leqslant 2$, or
- $G = \mathrm{M}_{23}$, $|x| = 23$ and $d(x, \mathrm{Inv}(G)) = 3$.

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Suppose $G = \mathrm{M}_{23}$ and $|x| = 23$. Let $B_r(x)$ be the ball of radius r at x.

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- If $J = \mathrm{M}_{11}$ or M_{22}, then $N_J(\langle y \rangle) = C_{11}{:}C_5$ is the only maximal soluble subgroup of J containing y, so $\{|z| \,:\, z \in B_2(x)\} = \{5, 11, 23\}$.

First reductions

Lemma. Let G be a finite insoluble group. Then

- $\Gamma_{\mathcal{S}}(G)$ is connected iff $\Gamma_{\mathcal{S}}(G/R(G))$ is connected.
- In addition, $d_{\mathcal{S}}(G) = d_{\mathcal{S}}(G/R(G))$.

Proof. $\langle x, y\rangle \leqslant G$ is soluble iff $\langle xR(G), yR(G)\rangle \leqslant G/R(G)$ is soluble. $\square$

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- This leaves the **monolithic** case ($\mathrm{soc}(G) = T \times \cdots \times T$, T simple).

Symmetric groups

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  By **B, Guralnick & Saxl (2011)**, there exists $g \in G$ with $H \cap H^g = 1$, hence $B_1(x) \cap B_1(x^g) = \emptyset$ and thus $d(x, x^g) \geqslant 3$.

Alternating groups

Theorem. Let $G = A_n$ with $n \geqslant 6$.

- We have $3 \leqslant d_{\mathcal{S}}(G) \leqslant 5$.
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Let $\mathcal{A}$ be the set of n-cycles in G and fix $x \in \mathcal{A}$. It suffices to show that

$$|\mathcal{A} \cap B_3(x)| = 1 + \alpha_1 + \alpha_2 + \alpha_3 < |\mathcal{A}|,$$

where $\alpha_k = |\{y \in \mathcal{A} \,:\, d(x,y) = k\}|$.

Sporadic groups

Theorem. Let G be an almost simple sporadic group with socle T.

- We have $3 \leqslant d_{\mathcal{S}}(G) \leqslant 5$.
- $d_{\mathcal{S}}(G) \geqslant 4$ only if $G = T$.
- $d_{\mathcal{S}}(G) = 4$ if $G = \mathrm{M}_{12}$, M_{22}, M_{23}, M_{24}, HS or J_3.
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The proof relies heavily on **computational methods** (using Magma).

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By random search, there exists $g \in G$ such that $B_1(x) \cap B_1(x^g) = \emptyset$ and $\langle a, b \rangle \notin \mathcal{S}$ for all $a \in B_1(x)$, $b \in B_1(x^g)$, so $d(x, x^g) \geqslant 4$.

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Therefore, $B_1(x) \cap B_1(y) \neq \emptyset$ for all $1 \neq x, y \in G$ and thus $d_{\mathcal{S}}(G) = 2$.

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Here $N_G(\langle z \rangle)$ is soluble and contains x and an element y of prime order s with $(T, s) \notin \mathcal{L}$. So $x \sim y$ and $d(y, \mathrm{Inv}(G)) = 1$ by the proposition. □

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Question. Can we determine the simple groups with $d_{\mathcal{S}}(G) = 2$?

Here we can prove that G has to be a **classical** group. The only known examples are $\mathrm{PSL}_2(q)$ with $q \geqslant 4$ even, $\mathrm{PSL}_3(2)$ and $\mathrm{PSU}_4(2)$.

Generalisations

Let $\mathcal{F}$ be a family of groups (e.g. abelian, soluble, nilpotent, metacyclic, metabelian, etc.) and fix a finite group $G \notin \mathcal{F}$.

Let $\Lambda_{\mathcal{F}}(G)$ be the graph with vertex set G, where $x \sim y$ iff $\langle x, y \rangle \in \mathcal{F}$.

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Some immediate questions:

- Can we identify $\mathcal{U}_{\mathcal{F}}(G)$?
- If it is a subgroup, how are $\Gamma_{\mathcal{F}}(G)$ and $\Gamma_{\mathcal{F}}(G/\mathcal{U}_{\mathcal{F}}(G))$ related?

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e.g. if $\mathcal{F} = \mathcal{A}$ is the collection of abelian groups, then the answers are **no** to Q1, but **yes** to Q2 (with $c = 10$).

e.g. if $\mathcal{F} = \mathcal{S}$ or $\mathcal{N}$, then **yes** to Q1 (with $c = 5$ or 10, respectively).